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Separation of variables for the quantum $\mathrm{SL}(2,\mathbb{R})$ spin chain

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Abstract: We construct a representation of the Separated Variables (SoV) for the quantum $\mathrm{SL}(2,\mathbb{R})$ Heisenberg closed spin chain following the Sklyanin's approach and obtain the integral representation for the eigenfunctions of the model. We calculate explicitly the Sklyanin measure defining the scalar product in the SoV representation and demonstrate that the language of Feynman diagrams is extremely useful in establishing various properties of the model. The kernel of the unitary transformation to the SoV representation is described by the same "pyramid diagram" as appeared before in the SoV representation for the $\mathrm{SL}(2,\mathbb{C})$ spin magnet. We argue that this kernel is given by the product of the Baxter $\mathbb{Q}$-operators projected onto a special reference state.

Keywords: Discrete and Finite Symmetries, Integrable Hierarchies, Lattice Integrable Models, Bethe Ansatz.

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1. Introduction

In this paper we study the spectral problem for quantum SL$(2,\mathbb{R})$ spin magnet within the Quantum Inverse Scattering Method [1, 2, 3]. The model is a generalization of the famous spin$-1/2$ Heisenberg XXX chain. It describes the nearest neighbour interaction between N spins with the corresponding spin operators being the generators of infinite-dimensional representation of the SL$(2,\mathbb{R})$ group.

The interest to studying noncompact spin magnets is twofold. From the one side, there exists a deep relation between integrable models and quantum $(3+1)$-dimensional Yang-Mills theories [4]. It turns out that the scale dependence of certain correlation functions on the light-cone, like those defining a baryon distribution amplitude and twist-three light-cone distributions, is governed by the evolution equations which upon redefinition of the variables coincide with the Schrödinger equation for the SL$(2,\mathbb{R})$ spin chain [5, 6, 7]. The number of sites in the lattice model is equal to the number of fields entering the correlation functions. The spin operators are the generators of the so-called collinear SL$(2,\mathbb{R})$ subgroup of the full conformal group of the classical Yang-Mills lagrangian. Another example comes from the studies of high-energy (Regge) asymptotics of the scattering amplitudes in multi-colour

QCD. As was shown in [8, 9], the spectrum of multi-gluonic compound states responsible for a power rise with the energy of the scattering amplitudes is described by the $\mathrm{SL}(2,\mathbb{C})$ spin magnet.

From the other side, an exact solution of the spectral problem for completely integrable quantum mechanical systems with infinitely dimensional quantum space, like periodic Toda chain and noncompact SL(2) spin magnet, represents a challenge for the theory of Integrable Models. The conventional methods like the Algebraic Bethe Ansatz [1, 3] (ABA) are not always applicable to such systems and one has to rely instead on the methods of the Baxter $\mathbb{Q}$-operator [10] and the Separation of Variables (SoV) [11]. Being combined together, the two methods allow us to determine the energy spectrum of the model in terms of the eigenvalues of the $\mathbb{Q}$-operator and construct an integral representation for the corresponding eigenfunctions by going over to the representation of the separated variables.

In spite of the fact that the methods have been formulated a long time ago, the explicit construction of the $\mathbb{Q}$-operator and the unitary transformation to the SoV representation for each particular model remains an extremely nontrivial task (for some known examples see review [12] and references therein). It is equally nontrivial to solve the emerging functional relations (the Baxter equations) and reconstruct a fine structure of the spectrum.

A powerful algebraic approach to constructing the SoV representation has been developed by Sklyanin [11]. It allows one to establish the relations both for the eigenfunctions in the separated coordinates and for the scalar product in the SoV representation. Their solutions are defined up to multiplication by an arbitrary periodic function. For models with finite-dimensional quantum space, like conventional SU(2) Heisenberg magnet, the separated coordinates take a discrete, finite set of values and, as a consequence, the above relations can be uniquely solved. For models with infinite-dimensional quantum space, like $\mathrm{SL}(2,\mathbb{R})$ Heisenberg magnet and periodic Toda chain, the separated coordinates take continuous values and the question arises how to fix the ambiguity or, equivalently, what are the additional conditions that one has to impose on the eigenfunctions and the integration measure in the SoV representation. To answer this question within the Sklyanin's approach, one has to provide an explicit construction of the unitary transformation to the SoV representation and identify the analytical properties of the resulting expressions for the eigenfunctions in the separated coordinates.

For the periodic Toda chain this program has been carried out by Kharchev and Lebedev in the series of papers [13]. The approach proposed in ref. [13] is based on the relation between the wave functions of the periodic and open Toda chains originating from a specific form of the Lax operator. Notice that similar relation does not hold for the SL(2) magnet. In the present paper we construct the SoV representation for the $\mathrm{SL}(2,\mathbb{R})$ spin chain following the method developed in [14, 15] in application to the $\mathrm{SL}(2,\mathbb{C})$ spin magnet. The method is general enough as it is applicable to both the $\mathrm{SL}(2,\mathbb{R})$ spin chain and the Toda chain.

Our main results include: calculation of the Sklyanin's measure defining the scalar product in the SoV representation, establishing the relation between the kernel of the unitary transformation to the SoV representation and the Baxter $\mathbb{Q}$-operator constructed in [16], proof of the equivalence of the ABA and the SoV method for the $\mathrm{SL}(2,\mathbb{R})$ spin chain.

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The paper is organized as follows. In section 2 we define the model and describe its integrability properties. In section 3 we apply the Sklyanin approach and present an explicit construction of the unitary transformation to the Separated Variables for the $\mathrm{SL}(2,\mathbb{R})$ closed spin chain. We demonstrate that the kernel of the SoV transformation admits a simple interpretation in terms of Feynman diagrams. In section 4 we follow the diagrammatical approach and derive the invariant scalar product in the SoV representation. In section 5 we establish the relation between the transition function to the SoV representation and the Baxter $\mathbb{Q}$-operator. It allows us to obtain the expressions for the eigenfunctions of the model in the separated variables. Section 5 contains concluding remarks. Some details of the calculations can be found in appendix A. In appendix B we argue that the Algebraic Bethe Ansatz and the SoV method lead to the same expressions for the eigenstates of the $\mathrm{SL}(2,\mathbb{R})$ spin magnet.

2. The quantum $\mathrm{SL}(2,\mathbb{R})$ spin magnet

The quantum $\mathrm{SL}(2,\mathbb{R})$ spin magnet is a one-dimensional lattice model of N interacting spins $\vec{S}_n = (S_n^0\,, S_n^+\,, S_n^-)$ (with $n = 1,\ldots,N$). The spin operators in different sites commute with each other and obey the standard $sl(2)$ commutation relations

$$[S_n^0, S_n^{\pm}] = \pm\,\hbar\, S_n^{\pm}\,, \qquad [S_n^+, S_n^-] = 2\hbar\, S_n^0\,. \tag{2.1}$$

The corresponding quadratic Casimir operator is defined as

$$\vec{S}_n^2 = S_n^0 S_n^0 + \frac{1}{2}(S_n^+ S_n^- + S_n^- S_n^+) = \hbar^2\, s_n(s_n-1)\,. \tag{2.2}$$

We shall impose periodic boundary conditions, $\vec{S}_{N+1} \equiv \vec{S}_1$, and put $\hbar = 1$ for simplicity. In addition, we shall assume that the spin chain is homogenous, $s_1 = \cdots = s_N = s$. Generalization to the case of inhomogeneous spin chain will be discussed in section 6.

2.1 Hamiltonian of the model

The hamiltonian of the homogenous $\mathrm{SL}(2,\mathbb{R})$ spin magnet is defined as [2, 17]

$$\mathcal{H}_N = \sum_{n=1}^{N} H_{n,n+1}\,, \qquad H_{n,n+1} = \psi(J_{n,n+1}) - \psi(2s)\,, \tag{2.3}$$

where $\psi(z) = d\ln\Gamma(z)/dz$ is the Euler ψ-function. The operator $J_{n,n+1}$ is related to the sum of two neighbouring spins

$$J_{n,n+1}(J_{n,n+1}-1) = (\vec{S}_n + \vec{S}_{n+1})^2\,, \tag{2.4}$$

$J_{N,N+1} = J_{N,1}$ and its eigenvalues satisfy the condition $J_{n,n+1} \geq 1/2$. The hamiltonian (2.3) has been constructed in ref. [2] as a generalization of the spin$-1/2$ XXX Heisenberg spin chain to high-dimensional representations of the SU(2) group. By the construction, $\mathcal{H}_N$ possesses a set of mutually commuting integrals of motion that we shall denote as $\boldsymbol{q} = (q_2,\ldots,q_N)$. Their number is large enough for the model to be completely integrable. The definition of the operators $\boldsymbol{q}$ will be given below (see eq. (2.18)).

In what follows, we shall use a particular representation for the spin operators

$$S_n^+ = iz_n^2 p_n + 2s\, z_n\,, \qquad S_n^- = -ip_n\,, \qquad S_n^0 = iz_n p_n + s\,, \tag{2.5}$$

where $p_n = -i\,\partial/\partial z_n$ and $[x_n, p_m] = i\,\delta_{nm}$. The spin s is assumed to be real and $s \geq 1/2$. In this representation, the $\mathrm{SL}(2,\mathbb{R})$ spin magnet (2.3) can be interpreted as a one-dimensional quantum mechanical model of N interacting particles with the coordinates z_n and the conjugated momenta p_n $(n = 1, \ldots, N)$. At $N = 3$ this model has appeared in high-energy QCD as describing the spectrum of the anomalous dimensions of the baryon distribution amplitudes [6].

The spin operators (2.5) act on the Hilbert space, V_n, of functions $\Psi(z_n) \in V_n$ holomorphic in the upper half-plane,[1] $\mathrm{Im}\, z_n > 0$, and normalizable with respect to the scalar product [18]

$$\|\Psi\|^2 = \int_{\mathrm{Im}\, z>0} \mathcal{D}z\, |\Psi(z)|^2\,, \tag{2.6}$$

with $z = x+iy$. Here integration is performed over the upper half-plane and the integration measure isdefined as[2]

$$\mathcal{D}z = \frac{2s-1}{\pi}\, d^2z\, (2\,\mathrm{Im}\, z)^{2s-2} = \frac{2s-1}{\pi} dxdy\, (2y)^{2s-2}\,. \tag{2.7}$$

The spin s in eqs. (2.5) and (2.7) takes arbitrary real values $s \geq 1/2$. For s integer or half integer, the Hilbert space V_n coincides with the linear space of the unitary irreducible representation of the $\mathrm{SL}(2,\mathbb{R})$ group of the discrete series [18]

$$\left[T(g^{-1})\Psi\right](z) = \frac{1}{(cz+d)^{2s}}\, \Psi\left(\frac{az+b}{cz+d}\right)\,, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\,.$$

In this paper we shall study the Schrödinger equation for the hamiltonian (2.3)

$$\mathcal{H}_N \Psi_{\boldsymbol{q}}(z_1, \ldots, z_N) = E_{\boldsymbol{q}}\, \Psi_{\boldsymbol{q}}(z_1, \ldots, z_N)\,. \tag{2.8}$$

The hamiltonian $\mathcal{H}_N$ acts on the quantum space of the model $\mathcal{V}_N$, given by the direct product of the Hilbert spaces in each site, $\mathcal{V}_N = \prod_{n=1}^N \otimes V_n$. In distinction with conventional (compact) Heisenberg spin magnet, $\mathcal{V}_N$ is infinite-dimensional for arbitrary finite N. The eigenstates $\Psi_{\boldsymbol{q}}(z_1, \ldots, z_N) \in \mathcal{V}_N$ are holomorphic functions of the z-coordinates in the upper half-plane, $\mathrm{Im}\, z_n > 0$ for $n = 1, \ldots, N$, normalizable with respect to the scalar product

$$\|\Psi_N\|^2 = \int \mathcal{D}^N z\, |\Psi(z_1, \ldots, z_N)|^2\,. \tag{2.9}$$

[1] One can choose instead $\Psi(z_n)$ to be holomorphic in the lower half-plane. As we will show below, the two cases, $\mathrm{Im}\, z > 0$ and $\mathrm{Im}\, z < 0$, correspond to the different values of the total momentum of the system, $p > 0$ and $p < 0$, respectively.

[2] Performing the conformal mapping $w = i(z-i)/(z+i)$, one can bring this expression to a canonical form involving the integration over an interior of the unit disk in the w-plane [18].

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Here $\int \mathcal{D}^N z = \prod_{n=1}^N \int_{\mathrm{Im}\, z_n>0} \mathcal{D} z_n$ and the measure $\mathcal{D} z_n$ is given by (2.7). In eq. (2.8) we indicated explicitly the dependence on the integrals of motion $\boldsymbol{q} = (q_2, \ldots, q_N)$. The hamiltonian (2.3) commutes with the total spin of the magnet $\vec{S} = \vec{S}_1 + \cdots + \vec{S}_N$ and one of its component, $iS_- = p_1 + \cdots + p_N$, defines the total momentum of the system. This allows us to assign a definite value of the momentum to the solutions to (2.8), $iS_- \Psi_{\boldsymbol{q},p} = p\, \Psi_{\boldsymbol{q},p}$, leading to

$$\Psi_{\boldsymbol{q},p}(z_1, \ldots, z_N) = \int_{-\infty}^{\infty} dx_0\ \mathrm{e}^{ip\, x_0}\, \Psi_{\boldsymbol{q}}(z_1 - x_0, \ldots, z_N - x_0)\,, \tag{2.10}$$

where integration goes along the real x-axis. Notice that in virtue of the SL(2) invariance of the hamiltonian, $[\mathcal{H}_N, \vec{S}] = 0$, the energy $E_{\boldsymbol{q}}$ does not depend on p.

It is straightforward to verify that for arbitrary real s the spin operators (2.5) are anti-hermitian with respect to the scalar product (2.6) and (2.9)

$$\left(S_n^0\right)^\dagger = -S_n^0\,, \qquad \left(S_n^\pm\right)^\dagger = -S_n^\pm \tag{2.11}$$

This property ensures that the hamiltonian (2.3) and the total momentum operator iS_- are hermitian on the Hilbert space of the model. As a consequence, the energy $E_{\boldsymbol{q}}$ and the total momentum p take real values. As we will show in section 3, the property (2.11) plays an important rôle in our construction of the SoV representation.

2.2 Integrals of motion

To construct the integrals of motion of the model, one follows the R-matrix approach [1, 3, 2]. The Lax operator for the $\mathrm{SL}(2,\mathbb{R})$ magnet is defined as

$$L_n(u) = u + i(\vec{\sigma} \cdot \vec{S}_n) = \begin{pmatrix} u + iS_n^0 & iS_n^- \\ iS_n^+ & u - iS_n^0 \end{pmatrix}. \tag{2.12}$$

It acts on the direct product of the auxiliary space and the quantum space in the nth site, $\mathbb{C}^2 \otimes V_n$, and satisfies the Yang-Baxter commutation relations involving a rational R-matrix [1, 3, 2]. Taking the product of N Lax operators in the auxiliary space, one obtains the monodromy matrix

$$\mathbb{T}_N(u) = L_1(u) \ldots L_N(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix}. \tag{2.13}$$

with the operators $A_N(u)$, …, $D_N(u)$ acting on the quantum space of the model $\mathcal{V}_N$. They satisfy the following Yang-Baxter relations [1]

$$\begin{aligned} B_N(u)\, B_N(v) &= B_N(v)\, B_N(u)\,, \\ (v - u + i)\, A_N(v)\, B_N(u) &= (v-u)\, B_N(u)\, A_N(v)\ +\ i\, A_N(u)\, B_N(v)\,, \\ (v - u - i)\, D_N(v)\, B_N(u) &= (v-u)\, B_N(u)\, D_N(v)\ -\ i\, D_N(u)\, B_N(v)\,. \end{aligned} \tag{2.14}$$

The quantum determinant of the monodromy matrix (2.13) is given by [19]

$$\det{}_{\mathrm{q}} \mathbb{T}_N(u) = A_N(u)\, D_N(u+i) - C_N(u)\, B_N(u+i) = (u+is)^N\, (u+i-is)^N\,. \tag{2.15}$$

It follows from the Yang-Baxter relations that the auxiliary transfer matrix $\widehat{t}_N(u)$, defined as a trace of the monodromy matrix over the auxiliary space

$$\widehat{t}_N(u) = \mathrm{tr}\ \mathbb{T}_N(u) = A_N(u) + D_N(u)\,, \tag{2.16}$$

commutes with itself for different values of the spectral parameter, with the hamiltonian of model (2.3) and the operator of the total spin

$$\left[\widehat{t}_N(u), \widehat{t}_N(v)\right] = \left[\widehat{t}_N(u), \mathcal{H}_N\right] = \left[\widehat{t}_N(u), \vec{S}\right] = 0\,. \tag{2.17}$$

Substituting (2.13) into (2.16) and taking into account the explicit form of the Lax operator (2.12) one finds that $\widehat{t}_N(u)$ is a polynomial of degree N in the spectral parameter u with the operator valued coefficients

$$\widehat{t}_N(u) = 2u^N + \widehat{q}_2\, u^{N-2} + \ldots + \widehat{q}_N\,. \tag{2.18}$$

One deduces from (2.17) that the operators $\widehat{q}_2$, …, $\widehat{q}_N$ form a family of mutually commuting, SL(2) invariant integrals of motion. Together with the total momentum of the system p, their eigenvalues $\boldsymbol{q} = (q_2, \ldots, q_N)$ form a complete set of the quantum numbers specifying the solutions to the Schrödinger equation (2.8). As a consequence, the spectral problem (2.8) can be reformulated as

$$\widehat{t}_N(u)\, \Psi_{\boldsymbol{q},p}(z_1, \ldots, z_N) = t_N(u)\, \Psi_{\boldsymbol{q},p}(z_1, \ldots, z_N)\,, \tag{2.19}$$

with $t_N(u)$ being an eigenvalue of the transfer matrix (2.18). The hamiltonian of the model, eq. (2.3), can be obtained in a similar manner from the fundamental transfer matrix, which is constructed analogously to (2.16) and (2.13) from the Lax operators acting on the direct product of two copies of the quantum space $V \otimes V$ [2].

As follows from their definition, eqs. (2.18), (2.16) and (2.12), the integrals of motion $\widehat{q}_k$ take a form of polynomials in the spin operators $\vec{S}_n$ (with $n = 1, \ldots, N$) of degree k. For instance,

$$\widehat{q}_2 = -2 \sum_{m>n} (\vec{S}_m \cdot \vec{S}_n) = -\vec{S}^2 + Ns(s-1)\,, \tag{2.20}$$

with $\vec{S}^2$ being the Casimir operator corresponding to the total spin of N particles. In the representation (2.5), $\widehat{q}_k$ is given by a complicated kth order differential operator and eq. (2.19) leads to the system of $(N-1)$-differential equations $(\widehat{q}_k - q_k)\Psi_{\boldsymbol{q},p}\,(z_1, \ldots, z_N) = 0$ with $k = 2, \ldots, N$. Its exact solution for arbitrary N becomes problematic. To overcome this difficulty we shall apply the method of Separated Variables (SoV) developed by Sklyanin in [11].

3. Separation of Variables for the $\mathrm{SL}(2, \mathbb{R})$ magnet

Let us construct an integral representation for the eigenfunctions of the quantum $\mathrm{SL}(2, \mathbb{R})$ magnet, $\Psi_{\boldsymbol{q},p}\,(z_1, \ldots, z_N)$, by going over from the coordinate z-representation to the representation of the Separated Variables $(p, \boldsymbol{x}) = (p, x_1, \ldots, x_{N-1})$

$$\Psi_{\boldsymbol{q},p}\,(z_1, \ldots, z_N) = \int d^{N-1}\boldsymbol{x}\, \mu(\boldsymbol{x})\, U_{p,\boldsymbol{x}}(z_1, \ldots, z_N)\, \Phi_{\boldsymbol{q}}(\boldsymbol{x})\,, \tag{3.1}$$

where $\mu(\boldsymbol{x})$ is the integration measure on the $\boldsymbol{x}$-space and integration region will be specified below. $U_{p,\boldsymbol{x}}(z_1,\ldots,z_N)$ is the kernel of the unitary operator corresponding to this transformation

$$U_{p,\boldsymbol{x}}(z_1,\ldots,z_N) = \langle z_1,\ldots,z_N|p,\boldsymbol{x}\rangle\,, \tag{3.2}$$

where we introduced standard notations for the bra- and ket-vectors on the quantum space of the model. $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$ is the eigenfunction in the separated coordinates

$$\Phi_{\boldsymbol{q}}(\boldsymbol{x})\,\delta(p-p') = \langle p',\boldsymbol{x}|\Psi_{\boldsymbol{q},p}\rangle = \int \mathcal{D}^N z\,\left(U_{p',\boldsymbol{x}}(z_1,\ldots,z_N)\right)^*\Psi_{\boldsymbol{q},p}\,(z_1,\ldots,z_N)\,. \tag{3.3}$$

A unique feature of the SoV representation is that the eigenfunction $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$ is factorized into a product of functions depending on a single variable $\Phi_{\boldsymbol{q}}(\boldsymbol{x}) \sim Q(x_1)\ldots Q(x_{N-1})$.

In this section, we shall obtain an explicit expression for the transition function to the SoV transformation, $U_{p,\boldsymbol{x}}(z_1,\ldots,z_N)$. The integration measure $\mu(\boldsymbol{x})$ and the properties of the eigenfunctions $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$ will be discussed in sections 4 and 5, respectively.

3.1 Basics of the SoV method

To construct the unitary transformation to the SoV representation, eq. (3.2), we apply the Sklyanin's approach [11]. In this approach, the basis vectors $|p,\boldsymbol{x}\rangle$ entering (3.2) are defined as eigenstates of the operator $B_N(u)$. We recall that $B_N(u)$ was introduced in (2.13) as an off-diagonal component of the monodromy operator. From the Yang-Baxter relations for the monodromy operator follows that $[B_N(u),B_N(v)] = [B_N(u),S_-] = 0$. This allows one to define the eigenfunctions of the operator $B_N(u)$ in such a way that they are u-independent and diagonalize the operator of the total momentum $(iS_- - p)|p,\boldsymbol{x}\rangle = 0$.

One finds from (2.13) and (2.12) that $B_N(u)$ is a polynomial in u of degree $N-1$ with operator valued coefficients, $B_N(u) = iS_-\,u^{N-1}+\ldots$. It eigenvalues can be specified by the total momentum p and by the set of zeros $\boldsymbol{x} = (x_1,\ldots,x_{N-1})$

$$B_N(u)\,U_{p,\boldsymbol{x}}(z_1,\ldots,z_N) = p\,(u-x_1)\cdots(u-x_{N-1})\,U_{p,\boldsymbol{x}}(z_1,\ldots,z_N)\,. \tag{3.4}$$

Notice that this relation is equivalent to the following system of equations

$$iS_-\,U_{p,\boldsymbol{x}}(z) = p\,U_{p,\boldsymbol{x}}(z)\,,\qquad B_N(x_k)\,U_{p,\boldsymbol{x}}(z) = 0\,,\qquad (k=1,\ldots,N-1)\,, \tag{3.5}$$

where $z=(z_1,\ldots,z_N)$. The solutions to eqs. (3.4) and (3.5) are defined up to an overall normalization factor. It is convenient to choose it in such a way that $U_{p,\boldsymbol{x}}(z_1,\ldots,z_{N-1})$ will be symmetric under permutations of any pair of the separated coordinates

$$U_{p,\cdots x_n\cdots x_m\cdots}(z_1,\ldots,z_{N-1}) = U_{p,\cdots x_m\cdots x_n\cdots}(z_1,\ldots,z_{N-1})\,. \tag{3.6}$$

Using (2.12), (2.13) and (2.11), one can verify that $B_N(u)$ is a self-adjoint operator on the Hilbert space of the model for real u

$$(B_N(u))^\dagger = B_N(u^*)\,. \tag{3.7}$$

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This implies that the parameters $\boldsymbol{x}$ parameterizing its eigenvalues, eq. (3.4), take *continuous real* values. The set of corresponding eigenstates $|p,\boldsymbol{x}\rangle$ is complete on the quantum space of the model $\mathcal{V}_N$ and their orthogonality condition looks as

$$\begin{aligned}\langle p',\boldsymbol{x'}|p,\boldsymbol{x}\rangle &= \int \mathcal{D}^N z\, U_{p,\boldsymbol{x}}(z_1,\dots,z_N)\big(U_{p',\boldsymbol{x'}}(z_1,\dots,z_N)\big)^* \\ &= \delta(p-p')\,\{\delta(\boldsymbol{x}-\boldsymbol{x'})+\cdots\}\,\frac{\mu^{-1}(\boldsymbol{x})}{(N-1)!}\,,\end{aligned} \tag{3.8}$$

where $\delta(\boldsymbol{x}-\boldsymbol{x'})\equiv\prod_{k=1}^{N-1}\delta(x_k-x'_k)$ and ellipses denote the sum of terms with all possible permutations inside the set $\boldsymbol{x}=(x_1,\dots,x_{N-1})$.

The property (3.7) also holds for the operators $A_N(u)$, $D_N(u)$, $C_N(u)$ and for the transfer matrix, $(\widehat{t}_N(u))^\dagger=\widehat{t}_N(u^*)$. Applying the both sides of the Yang-Baxter relations (2.14) to $U_{p,\boldsymbol{x}}(z_1,\dots,z_N)$ and taking $v=x_k$, one finds

$$A_N(x_k)\,U_{p,\boldsymbol{x}}(z)=a_k(\boldsymbol{x})\,U_{p,\boldsymbol{x}+i\boldsymbol{e}_k}(z)\,,\qquad D_N(x_k)\,U_{p,\boldsymbol{x}}(z)=d_k(\boldsymbol{x})\,U_{p,\boldsymbol{x}-i\boldsymbol{e}_k}(z_i)\,. \tag{3.9}$$

Here the notation was introduced for the unit vectors $\boldsymbol{e_k}$ in the $(N-1)$-dimensional $\boldsymbol{x}$-space, such that $(\boldsymbol{e_k})_n=\delta_{nk}$ and $\boldsymbol{x}+i\boldsymbol{e}_k\equiv(x_1,\dots,x_k+i,\dots,x_{N-1})$. Substituting $u=x_k$ into (2.15) and applying (3.9), one finds that $a_k(\boldsymbol{x})\,d_k(\boldsymbol{x}+i\boldsymbol{e}_k)=(x_k+is)^N\,(x_k+i-is)^N$. The coefficients $a_k(\boldsymbol{x})$ and $d_k(\boldsymbol{x})$ depend on the normalization of $U_{p,\boldsymbol{x}}(z)$, or equivalently on the definition of the integration measure in (3.8). It is convenient to normalize $U_{p,\boldsymbol{x}}(z)$ in such a way that

$$a_k(\boldsymbol{x})=\Delta_+(x_k)=(x_k+is)^N\,,\qquad d_k(\boldsymbol{x})=\Delta_-(x_k)=(x_k-is)^N\,. \tag{3.10}$$

In eq. (3.9) we have tacitly assumed that the function $U_{p,\boldsymbol{x}}(z)$ can be analytically continued from real $\boldsymbol{x}$ into a finite strip in the complex plane. We will verify this property *a posteriori.*

Using the solutions to (3.4) and (3.5), one can decompose an arbitrary state on $\mathcal{V}_N$ over the basis of the functions $U_{p,\boldsymbol{x}}(z_1,\dots,z_N)$. For the eigenstates of the model, the decomposition takes the form (3.1) and (3.3). To obtain the wave function in the separated coordinates, $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$, one has to examine the action of the transfer matrix on the function $U_{p,\boldsymbol{x}}(z_1,\dots,z_N)$. According to (2.18), $\widehat{t}_N(u)$ is a polynomial of degree N in u with the coefficient in front of u^{N-1} equal to zero. As a result, it can be reconstructed from its values at $N-1$ distinct points $\widehat{t}_N(x_k)=A_N(x_k)+D_N(x_k)$, with $k=1,\dots,N-1$, using the Lagrange interpolation formula. Applying (3.9) one gets

$$\begin{aligned}\widehat{t}_N(u)\,U_{p,\boldsymbol{x}}(z) &= 2\left(u+\sum_{k=1}^{N-1}x_k\right)\prod_{j=1}^{N-1}(u-x_j)U_{p,\boldsymbol{x}}(z)+ \\ &+\sum_{k=1}^{N-1}\prod_{j\neq k}\frac{u-x_j}{x_k-x_j}\Big[\Delta_+(x_k)U_{p,\boldsymbol{x}+i\boldsymbol{e_k}}(z)+\Delta_-(x_k)U_{p,\boldsymbol{x}-i\boldsymbol{e_k}}(z)\Big]\,.\end{aligned} \tag{3.11}$$

The wave function (3.1) has to diagonalize the transfer matrix, eq. (2.19). Taking into account eq. (3.11), we find that the wave function in the separated coordinates satisfies a multi-dimensional Baxter equation

$$t_N(x_k)\,\Phi_{\boldsymbol{q}}(\boldsymbol{x})=(x_k+is)^N\,\Phi_{\boldsymbol{q}}(\boldsymbol{x}+i\boldsymbol{e}_k)+(x_k-is)^N\,\Phi_{\boldsymbol{q}}(\boldsymbol{x}-i\boldsymbol{e}_k)\,, \tag{3.12}$$

with $k = 1, \ldots, N-1$. Here we assumed that the integration contour in (3.1) can be shifted into the complex $\boldsymbol{x}$-plane. In addition, we took into account that the measure $\mu(\boldsymbol{x})$ satisfies the following finite-difference equation [11]

$$\frac{\mu(\boldsymbol{x}+i\boldsymbol{e_k})}{\mu(\boldsymbol{x})} = \frac{\Delta_+(x_k)}{\Delta_-(x_k+i)} \prod_{j\neq k} \frac{x_k - x_j + i}{x_k - x_j}\,, \tag{3.13}$$

with $\Delta_\pm(x_k)$ defined in (3.10). This relation follows from the condition for $\widehat{t}_N(u)$ to be a self-adjoint operator in the SoV representation, $\langle p', \boldsymbol{x}'|\widehat{t}_N(u)|p,\boldsymbol{x}\rangle^* = \langle p,\boldsymbol{x}|\widehat{t}_N(u^*)|p',\boldsymbol{x}'\rangle$.

The same Baxter equation (3.12) holds for a complex conjugated function

$$\Phi^*_{\boldsymbol{q}}(\boldsymbol{x})\,\delta(p-p') = \langle \Psi_{\boldsymbol{q},p}|p',\boldsymbol{x}\rangle\,. \tag{3.14}$$

To see this one uses (2.19) together with hermiticity of the transfer matrix

$$t_N(x_k)\langle \Psi_{\boldsymbol{q},p}|p',\boldsymbol{x}\rangle = \langle \widehat{t}_N(x_k)\Psi_{\boldsymbol{q},p}|p',\boldsymbol{x}\rangle = \langle \Psi_{\boldsymbol{q},p}|(A_N(x_k)+D_N(x_k)|p',\boldsymbol{x}\rangle\,. \tag{3.15}$$

Here in the last relation $A_N(x_k)$ and $D_N(x_k)$ act on $|p',\boldsymbol{x}\rangle$ as shift operators, eq. (3.9). Using (3.14), one finds that $\Phi^*_{\boldsymbol{q}}(\boldsymbol{x})$ satisfies (3.12) and, therefore, one should expect that $\Phi^*_{\boldsymbol{q}}(\boldsymbol{x}) \sim \Phi_{\boldsymbol{q}}(\boldsymbol{x})$ for real $\boldsymbol{x}$. We shall verify this relation below (see eqs. (5.17) and (5.8)).

As was mentioned in the Introduction, eqs. (3.13) and (3.12) do not allow us to uniquely determine the integration measure and the eigenfunctions in the SoV representation since their solutions are defined up to a multiplication by an arbitrary periodic function $f(x) = f(x\pm i)$. To fix the ambiguity one has to impose the additional conditions on the solutions to (3.13) and (3.12). As we will show in the next section, these conditions follow from the explicit expression for the kernel of the unitary transformation to the SoV representation, $U_{p,\boldsymbol{x}}(z)$.

3.2 Construction of the SoV representation

To determine the kernel $U_{p,\boldsymbol{x}}(z)$ one has to solve the system (3.5) under the additional condition (3.6). Constructing solution for $U_{p,\boldsymbol{x}}(z)$, we shall follow the approach developed in ref. [14] in application to the quantum $\mathrm{SL}(2,\mathbb{C})$ spin magnet.

To begin with, one considers the first equation in the system (3.5)

$$B_N(x_1)U_{p,\boldsymbol{x}}(z_1,\ldots,z_N) = 0\,. \tag{3.16}$$

Its solution can be found by making use of the invariance of the transfer matrix (2.16) under local gauge transformations of the Lax operators (2.12) and the monodromy matrix (2.13) [20, 16]

$$L_k(u) \to \widetilde{L}_k(u) = M_k^{-1}\,L_k(u)\,M_{k+1}\,, \qquad \mathbb{T}_N(u) \to \widetilde{\mathbb{T}}_N(u) = M_1^{-1}\,\mathbb{T}_N(u)\,M_1\,. \tag{3.17}$$

Here M_k are arbitrary 2×2 matrices, such that $M_{N+1} = M_1$ and $\det M_k \neq 0$. Let us choose the matrices M_k as

$$M_k = \begin{pmatrix} 1 & \bar{w}_k^{-1} \\ 0 & 1 \end{pmatrix}\,, \qquad M_k^{-1} = \begin{pmatrix} 1 & -\bar{w}_k^{-1} \\ 0 & 1 \end{pmatrix} \tag{3.18}$$

with $\bar{w}_1, \ldots, \bar{w}_N$ being arbitrary gauge parameters. It is straightforward to verify that being applied to the function

$$\phi_u(z_k; \bar{w}_k, \bar{w}_{k+1}) = (z_k - \bar{w}_k)^{-s-iu} (z_k - \bar{w}_{k+1})^{-s+iu}, \tag{3.19}$$

the gauge rotated Lax operator $\widetilde{L}_k(u)$ takes the form of a lower triangle matrix

$$\widetilde{L}_k(u) \cdot \phi_u(z_k; \bar{w}_k, \bar{w}_{k+1}) = \begin{pmatrix} (u+is)\phi_{u+i} & 0 \\ * & (u-is)\phi_{u-i} \end{pmatrix}. \tag{3.20}$$

This suggests to define the following function

$$Y_u(z, \bar{w}) = \prod_{k=1}^{N} \phi_u(z_k; \bar{w}_k, \bar{w}_{k+1}) = \prod_{k=1}^{N} (z_k - \bar{w}_k)^{-s-iu} (z_k - \bar{w}_{k+1})^{-s+iu}. \tag{3.21}$$

Aside from the coordinates $z = (z_1, \ldots, z_N)$, it depends on auxiliary variables $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_N)$ and the spectral parameter u. Denoting the matrix elements of $\widetilde{\mathbb{T}}_N(u)$ as $\widetilde{A}_N(u), \ldots, \widetilde{D}_N(u)$ in the same manner as (2.13), one finds from (3.20) that $\widetilde{B}_N(u) \times Y_u(z, \bar{w}) = 0$ and

$$\widetilde{A}_N(u) Y_u(z, \bar{w}) = (u+is)^N Y_{u+i}(z, \bar{w}), \qquad \widetilde{D}_N(u) Y_u(z, \bar{w}) = (u-is)^N Y_{u-i}(z, \bar{w}). \tag{3.22}$$

As follows from (3.17), the operators $\widetilde{A}_N(u), \ldots, \widetilde{D}_N(u)$ are given by linear combinations of the operators $A_N(u), \ldots, D_N(u)$ with the coefficients depending on $1/\bar{w}_1$. For $\bar{w}_1 \to \infty$ the two sets of the operators coincide since $M_1 = \mathbb{1}$ and $\widetilde{\mathbb{T}}_N(u) = \mathbb{T}_N(u)$. In this limit, the r.h.s. of (3.21) scales as $Y_u(z, \bar{w}) \sim \bar{w}_1^{2s} \Lambda_u(z, \bar{w})$ with

$$\begin{aligned} \Lambda_u(z_1, \ldots, z_N | \bar{w}_2, \ldots, \bar{w}_N) &= (z_1 - \bar{w}_2)^{-s+iu} \times \\ &\times \left(\prod_{k=2}^{N-1} (z_k - \bar{w}_k)^{-s-iu} (z_k - \bar{w}_{k+1})^{-s+iu} \right) (z_N - \bar{w}_N)^{-s-iu}. \end{aligned} \tag{3.23}$$

One deduces from (3.22) that for arbitrary $\bar{w}_2, \ldots, \bar{w}_N$ the function $\Lambda_u(z, \bar{w})$ satisfies the relations

$$\begin{aligned} B_N(u) \Lambda_u(z, \bar{w}) &= 0, \\ A_N(u) \Lambda_u(z, \bar{w}) &= (u+is)^N \Lambda_{u+i}(z, \bar{w}), \\ D_N(u) \Lambda_u(z, \bar{w}) &= (u-is)^N \Lambda_{u-i}(z, \bar{w}). \end{aligned} \tag{3.24}$$

It proves extremely useful to translate the obtained expression for the Λ-function, eq. (3.23), into a language of Feynman diagrams [14]. Namely, one represents each factor $(z - \bar{w})^{-\alpha}$ in the r.h.s. of (3.23) by an arrow with the index α that starts in the point $\bar{w}$ and ends in z. In this way, the Λ-function can be represented as a Feynman diagram shown in figure 1.

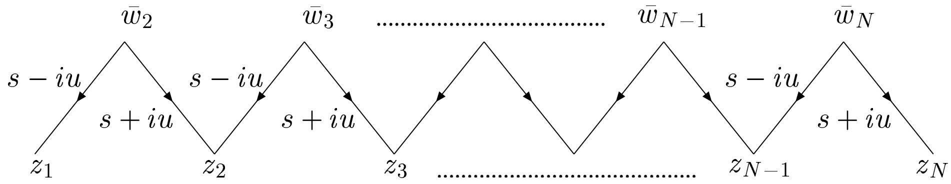

Figure 1: Diagrammatical representation of the function $\Lambda_u(z_1,\dots,z_N|w_2,\dots,w_N)$. The arrow with the index α that connects the points $\bar{w}$ and z stands for $(z-\bar{w})^{-\alpha}$.

Making use of the first relation in (3.24), one finds that the general solution to eq. (3.16) can be written as a convolution of the Λ-function with an arbitrary weight function $Z_{N-1}(w_2,\dots,w_N)$, which is holomorphic in the upper half-plane and depends, in general, on the separated variables $x_2,\dots,x_N$

$$\begin{aligned} U_{p,\boldsymbol{x}}(z_1,\dots,z_N) &= \int \mathcal{D}^{N-1}w\,\Lambda_{x_1}(z_1,\dots,z_N|\bar{w}_2,\dots,\bar{w}_N)\,Z_{N-1}(w_2,\dots,w_N) \\ &\equiv [\Lambda_N(x_1)\otimes Z_{N-1}](z_1,\dots,z_N)\,, \end{aligned} \tag{3.25}$$

where $\mathcal{D}^{N-1}w=\mathcal{D}w_2\dots\mathcal{D}w_N$ with $\bar{w}_n=w_n^*$. Here the notation was introduced for the integral operator $\Lambda_N(x_1)$ with the kernel defined by the function $\Lambda_{x_1}(z_1,\dots,z_N|\bar{w}_2,\dots,\bar{w}_N)$.

Let us now require that the function $U_{p,\boldsymbol{x}}(z)$ defined in (3.25) has to satisfy the second equation from the system (3.5), $B_N(x_2)\,U_{p,\boldsymbol{x}}(z)=0$. This leads to a rather complicated equation for the weight function $Z_{N-1}(w)$. To solve this equation, we shall propose a particular ansatz for $Z_{N-1}(w)$ and verify that it leads to the expression for $U_{p,\boldsymbol{x}}(z)$ which obeys (3.5). Suppose that there exists such $Z_{N-1}(w)$ that the resulting expression for (3.25) is a symmetric function of x_1 and x_2. Then, the above requirement will be automatically fulfilled in virtue of (3.16). The definition of $Z_{N-1}(w)$ is based on the following remarkable identity

$$[\Lambda_N(x_1)\otimes\Lambda_{N-1}(x_2)](z_1,\dots,z_N|v_3,\dots,v_N)=[\Lambda_N(x_2)\otimes\Lambda_{N-1}(x_1)](z_1,\dots,z_N|v_3,\dots,v_N)\,, \tag{3.26}$$

where the operator $\Lambda_N(x_2)$ was defined for arbitrary N in (3.25) and (3.23). The both sides of this relation can be rewritten as a convolution of two Λ-functions, eq. (3.23). In the Feynman diagram representation, the l.h.s. of (3.26) is described by the diagram shown in figure 2. Having the identity (3.26) in mind, we choose the function $Z_{N-1}(w)$ in the following form

$$Z_{N-1}(w_2,\dots,w_N)=[\Lambda_{N-1}(x_2)\otimes Z_{N-2}]\,(w_2,\dots,w_N)\,, \tag{3.27}$$

where the function $Z_{N-2}=Z_{N-2}(v_3,\dots,v_N)$ is holomorphic in the upper half-plane, $\mathrm{Im}\,v_k>0$, and depends, in general, on $x_3,\dots,x_N$. Substituting (3.27) into (3.25) and making use of (3.26) we find that $U_{p,\boldsymbol{x}}(z)$ is invariant under permutations of x_1 and x_2

The proof of (3.26) can be performed diagrammatically without doing any calculation. It is based on elementary permutation identities shown diagrammatically in figures 3 and 4. Their derivation can be found in appendix A. Similar identities have been also found for the Toda model [20] and the SL(2, $\mathbb{C}$) spin chain [14]. To prove (3.26) one inserts two auxiliary

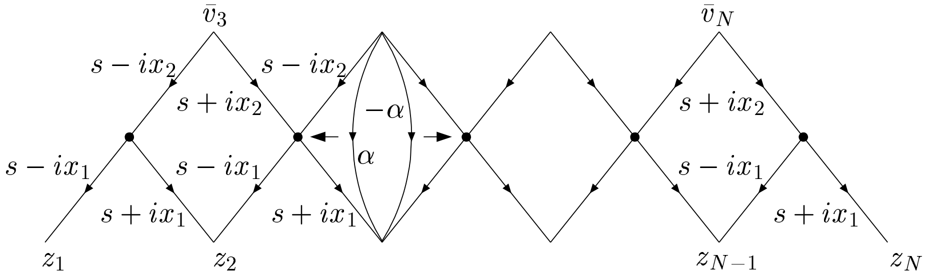

Figure 2: Diagrammatical proof of the identity (3.26). One inserts two vertical lines with the indices $\pm\alpha = \pm i(x_2 - x_1)$ into one of the rhombuses and displaces them in the directions indicated by arrows with a help of identities shown in figures 3 and 4.

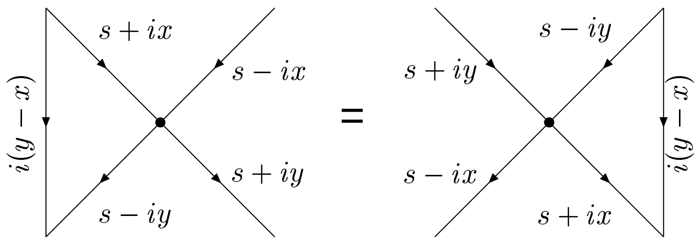

Figure 3: Permutation identity.

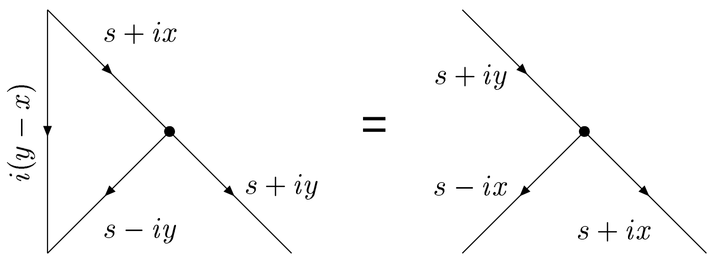

Figure 4: Special case of the permutation identity. It is obtained from figure 3 by sending one of the external points to infinity.

lines with the indices $\pm i(x_1 - x_2)$ into one of the rhombuses in figure 2. Since the lines are attached to the same points and the sum of their indices equals zero, this transformation does not affect the l.h.s. of (3.26). Then, one applies the permutation identity shown in figure 3 and moves the line with the index $i(x_2 - x_1)$ to the left part of the diagram until it reaches the leftmost rhombus in which case one applies the identity shown in figure 4. Performing similar transformations on the second line with the indices $-i(x_2 - x_1)$, one moves it to the right part of the diagram. In this way, one obtains the initial Feynman diagram but with the variables x_1 and x_2 interchanged, thus proving (3.26).

It is now straightforward to write a general expression for the function $U_{p,\boldsymbol{x}}(z)$ satisfying the relations (3.5) and (3.6)

$$U_{p,\boldsymbol{x}}(z_1,\dots,z_N) = p^{Ns-1/2} \int_{\mathrm{Im}\, w_N>0} \mathcal{D}w_N \; \mathrm{e}^{ip\, w_N}\; U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N)\,, \tag{3.28}$$

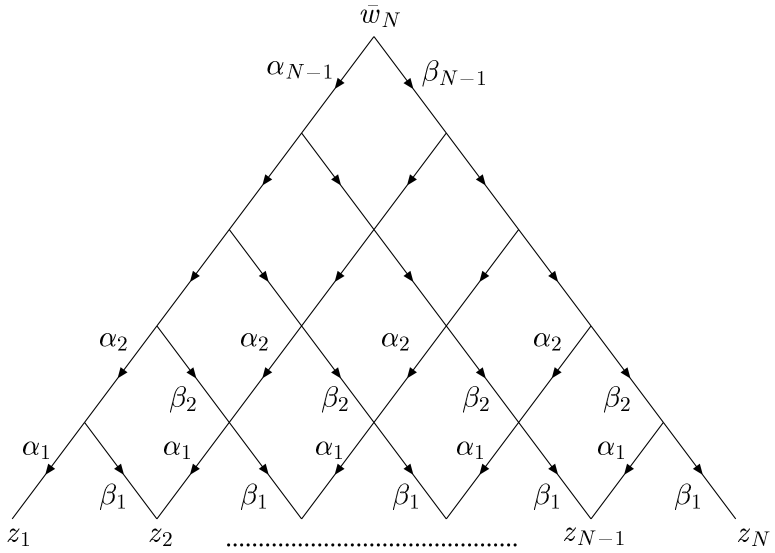

Figure 5: Diagrammatical representation of the function $U_{\boldsymbol{x}}(z;\bar{w}_N)$. The indices $\alpha_k = s - ix_k$ and $\beta_k = s + ix_k$ parameterize the corresponding factors entering (3.23). The SL$(2,\mathbb{R})$ integration (2.7) over the position of internal vertices is implied.

where $\vec{z} = (z_1, \dots, z_N)$ and $U_{\boldsymbol{x}}(\vec{z};\bar{w}_N)$ is factorized into the product of $N-1$ operators

$$U_{\boldsymbol{x}}(\vec{z};\bar{w}_N) = [\Lambda_N(x_1) \otimes \Lambda_{N-1}(x_2) \otimes \ldots \otimes \Lambda_2(x_{N-1})]\,(z_1, \dots, z_N|\bar{w}_N)\,. \tag{3.29}$$

Here the operator $\Lambda_n(x_{N+1-n})$ has an integral kernel $\Lambda_{x_{N+1-n}}(z_1, \dots, z_n|\bar{w}_2, \dots, \bar{w}_n)$ defined in (3.23). It depends on a single separated coordinate x_{N+1-n} (with $n = 2, \dots, N$) and being applied to an arbitrary function of $w_2, \dots, w_n$ it increases the number of its argument by 1 (see eq. (3.25)). The expression for the function $U_{\boldsymbol{x}}(\vec{z};\bar{w}_N)$, eq. (3.29), admits a simple diagrammatic representation. Replacing each Λ-operator in (3.29) by the corresponding Feynman diagram (see figure 1) one obtains that $U_{\boldsymbol{x}}(\vec{z};\bar{w}_N)$ can be represented as the "Pyramide du Louvre" diagram shown in figure 5. It consists of $(N-1)$-rows with each row denoting a single Λ-operator in (3.29). The k-th row of the pyramid consists of $(N-k)$-lines carrying the index $\alpha_k = s - ix_k$ and the same number of lines with the index $\beta_k = s + ix_k$ depending on the separated variable x_k. Remarkably enough, a pyramid diagram similar to the one shown in figure 5 describes the SoV transformation for the quantum SL$(2,\mathbb{C})$ spin chain [14, 15] and the periodic Toda chain [13]. We would like to stress that a pyramid-like form of $U_{\boldsymbol{x}}(\vec{z};\bar{w}_N)$ is a consequence of the factorized form of the kernel (3.29), which is expected to be a general feature of the SL(2) magnets and related quantum integrable models.

Let us verify that the obtained expression for the function $U_{\boldsymbol{x}}(\vec{z};\bar{w}_N)$, eq. (3.29), verifies the defining relations (3.5) and (3.6). Indeed, it follows from eqs. (3.26) and (3.16) that $U_{\boldsymbol{x}}(\vec{z};\bar{w}_N)$ is a completely symmetric function of $x_1, \dots, x_{N-1}$, which is nullified by the operators $B_N(x_k)$ for $k = 1, \dots, N-1$. The integration over w_N in (3.28) ensures that $U_{p,\boldsymbol{x}}(z)$ is an eigenfunction of the operator of the total momentum iS_-. To see this, one has to take into account that the kernel of the Λ-operator, eq. (3.23), depends on differences of the coordinates and, therefore, the function $U_{\boldsymbol{x}}(\vec{z};\bar{w}_N)$ is translation invariant, $U_{\boldsymbol{x}}(z+\epsilon;\bar{w}_N+\epsilon) = U_{\boldsymbol{x}}(z;\bar{w}_N)$ with ϵ real.[3] The normalization factor $p^{Ns-1/2}$ was inserted in the r.h.s. of (3.28) for the later convenience.

By the construction, $U_{\boldsymbol{x}}(\vec{z};\bar{w}_N)$ is a holomorphic function of $\vec{z}$ and $w_N = \bar{w}_N^*$ in the upper-half plane. One finds from (3.28) that $U_{p,\boldsymbol{x}}(z)$ vanishes for $p<0$ since the integration contour over $\mathrm{Re}\, w_N$ can be closed into the lower half-plane. Then, it follows from (3.1) that the eigenfunctions $\Psi_{\boldsymbol{q},p}(z)$ are different from zero only for $p>0$. This is a common feature of states belonging to the Hilbert space (2.9) and (2.6). The eigenstates with negative momenta, $p<0$, can be constructed on the space of functions holomorphic in the lower-half plane.

Since the kernel of the Λ-operator satisfies (3.24), one verifies that, in agreement with (3.9), the same relations hold for the function $U_{p,\boldsymbol{x}}(z)$. Using (2.16) one finds

$$\widehat{t}_N(x_k)\, U_{p,\boldsymbol{x}}(z) = (x_k+is)^N\, U_{p,\boldsymbol{x}+i\boldsymbol{e}_k}(z) + (x_k-is)^N\, U_{p,\boldsymbol{x}-i\boldsymbol{e}_k}(z)\,. \tag{3.30}$$

As was shown in section 3.1, this leads to the multi-dimensional Baxter equation for the wave function in the SoV representation, eq. (3.12). It follows from (3.28) and (3.29) that $U_{p,\boldsymbol{x}}(z)$ is an entire function of $x_1,\ldots,x_{N-1}$ and, as a consequence, the solutions to (3.12) have to possess the same property. Indeed, the analytical properties of $U_{p,\boldsymbol{x}}(z)$ are in one-to-one correspondence with possible divergences of the Feynman integral in the r.h.s. of (3.29). Using (3.23) one verifies that the integral is convergent for arbitrary complex $\boldsymbol{x}$.

3.3 Recurrence relations/Contour integral representation

Expressions (3.28) and (3.29) have a simple recursive form as a function of the number of particles N. Increasing this number, $N\to N+1$, one has just to add an additional row to the pyramid diagram in figure 5. Namely

$$\begin{aligned} &U_{x_1,\ldots,x_{N-1}}(z_1,\ldots,z_N;\bar{w}_N) = \\ &\qquad \int \mathcal{D}^{N-1}v\, \Lambda_{x_{N-1}}(z_1,\ldots,z_N|\bar{v}_1,\ldots,\bar{v}_{N-1})\, U_{x_1,\ldots,x_{N-2}}(v_1,\ldots,v_{N-1};\bar{w}_N)\,, \end{aligned} \tag{3.31}$$

with the Λ-function defined in (3.23). The $\mathrm{SL}(2,\mathbb{R})$ integrals over the v-coordinates can be simplified by making use of the identity (A.8). In this way, one gets

$$\begin{aligned} U_{x_1,\ldots,x_{N-1}}(z_1,\ldots,z_N;\bar{w}_N) &= \left[i^{2s}B(s+ix_{N-1},s-ix_{N-1})\right]^{-N+1}\prod_{k=1}^{N-1}\int_0^1 d\tau_k\,\tau_k^{s-ix_{N-1}-1}\times \\ &\times(1-\tau_k)^{s+ix_{N-1}-1}U_{x_1,\ldots,x_{N-2}}(z_1(\tau_1),\ldots,z_{N-1}(\tau_{N-1});\bar{w}_N)\,, \end{aligned} \tag{3.32}$$

where $B(x,y)$ is the Euler beta-function and $z_k(\tau_k) = (1-\tau_k)z_k+\tau_k z_{k+1}$, so that $z_k(0)=z_k$ and $z_k(1)=z_{k+1}$. The U-function in the r.h.s. is described by the same pyramid diagram but with one row less. Its end-points are located in between the end-points of the pyramid in the l.h.s. of (3.32). At $N=2$ one gets from (3.29)

$$U_{x_1}(z_1,z_2;\bar{w}_2) = (z_1-\bar{w}_2)^{-s+ix_1}(z_2-\bar{w}_2)^{-s-ix_1}\,. \tag{3.33}$$

[3] Notice that the $\mathrm{SL}(2,\mathbb{R})$ integration in eq. (3.28) can be replaced by the integration over the real axis $\int_{-\infty}^{\infty} dw_N$. According to (A.2), the result will differ by the normalization factor $p^{1-2s}\Gamma(2s)$.

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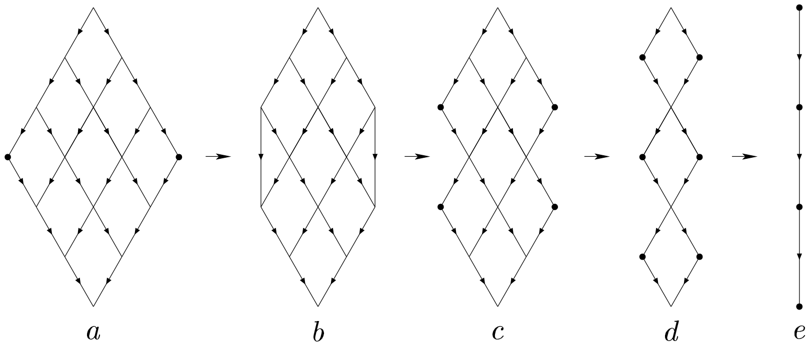

Figure 6: Diagrammatical calculation of the scalar product of two pyramids. The fat points indicate the vertices which can be integrated out using the chain relation shown in figure 7.

Repeatedly applying (3.32) and using (3.33) as a boundary condition, one can express $U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N)$ as a product of one-dimensional nested contour integrals. To save space, we do not present here the explicit expression.

4. Integration measure

To calculate the integration measure in the SoV representation, $\mu(\boldsymbol{x})$, one has to substitute the obtained expressions for the functions $U_{p,\boldsymbol{x}}(z_1,\ldots,z_N)$, eq. (3.28), into the orthogonality condition (3.8) and perform integration. In spite of the fact that a multi-dimensional integral in (3.8) seems to be rather complicated, it can be easily evaluated by making use of the diagrammatical representation for $U_{p,\boldsymbol{x}}(z)$ (see figure 5). The analysis goes along the same lines as calculation of the integration measure for the SL$(2,\mathbb{C})$ magnet [14].

Let us substitute (3.28) into (3.8) and consider the following scalar product

$$\langle w_N', \boldsymbol{x}' | w_N, \boldsymbol{x}\rangle = \int \mathcal{D}^N z \, U_{\boldsymbol{x}}(z_1,\ldots,z_N;\bar{w}_N)\big(U_{\boldsymbol{x}'}(z_1,\ldots,z_N;\bar{w}_N')\big)^*\,. \tag{4.1}$$

To evaluate (3.8) one has to perform a Fourier transformation of $\langle w_N', \boldsymbol{x}' | w_N, \boldsymbol{x}\rangle$ with respect to w_N and w_N' and multiply it by the additional factor p^{2Ns-1}. The function $U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N)$ entering (4.1) is represented by the pyramid diagram shown in figure 5. To obtain $(U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N'))^*$ one has to replace in eqs. (3.28) and (3.23) the holomorphic "propagators" $(z-\bar{w})^{-\alpha}$ by complex conjugated expressions $(\bar{z}-w)^{-\alpha^*} = \mathrm{e}^{i\pi\alpha^*}(w-\bar{z})^{-\alpha^*}$. The function $(U_{\boldsymbol{x}}(z;\bar{w}_N'))^*$ can be represented by the same pyramid diagram if one replaces in figure 5 the indices α_k and β_k by their conjugated expressions, $\alpha_k^* = \beta_k$ and $\beta_k^* = \alpha_k$, respectively, and flips the direction of all arrows. Each arrow is accompanied by the additional factor $\mathrm{e}^{i\pi\alpha_k^*}$ or $\mathrm{e}^{i\pi\beta_k^*}$. Combining these factors together along the rows of the pyramid diagram and using the identity $\alpha_k+\beta_k = 2s$, one finds that their total product equals $\mathrm{e}^{i\pi N(N-1)s}$. As we will show below, in the final expression for the measure this factor cancels against a similar factor coming from the z-integration in (4.1).

It is convenient to flip horizontally the conjugated pyramid diagram, so that the point w_N' will be located at the bottom of the diagram and the points $z_1,\ldots,z_N$ at the top. The scalar product (4.1) is obtained by sewing together the pyramid and its conjugated

Figure 7: Chain relation.

counterpart at the points $z_1, \ldots, z_N$. The resulting Feynman diagram takes the form of a big rhombus built out of $(N-1)^2$-elementary rhombuses (see the leftmost diagram in figure 6 below). Its tips have the coordinates $\bar{w}_N$ and w'_N. Notice that the indices α_k and β_k in the upper and lower part of this rhombus depend on two different sets of the separated coordinates $\boldsymbol{x}$ and $\boldsymbol{x}'$, respectively.

Let us first calculate (4.1) at $N = 2$. The corresponding rhombus diagram is shown in figure 8. Integration over z_1 and z_2 can be easily performed using the chain relation shown in figure 7 (see appendix A for details) leading to $\langle w'_2, x'_1 | w_2, x_1 \rangle \sim (w'_2 - \bar{w}_2)^0$. Its Fourier transformation with respect to w_2 and w'_2 leads to the expression for $\langle p', x'_1 | p, x_1 \rangle$, which is divergent at $p = p'$. This means that, in agreement with (3.8), $\langle p', x'_1 | p, x_1 \rangle$ should be understood as a distribution. To identify its form one has to regularize the corresponding Feynman integrals. As was shown in [14], this can be achieved by shifting the indices as

$$\alpha_k \to \alpha_k + \epsilon\,, \qquad \beta_k \to \beta_k + \epsilon \tag{4.2}$$

and carefully examining the limit $\epsilon \to 0$

$$\langle p', \boldsymbol{x}' | p, \boldsymbol{x} \rangle = (pp')^{Ns-1/2} \lim_{\epsilon \to 0} \int \mathcal{D}^N w_N \, \mathrm{e}^{ip\, w_N} \int \mathcal{D}^N w'_N \, \mathrm{e}^{-ip'\, \bar{w}_N} \langle w'_N, \boldsymbol{x}' | w_N, \boldsymbol{x} \rangle_\epsilon \,. \tag{4.3}$$

Repeating the calculation of the rhombus diagram in figure 8, one gets

$$\langle w'_2, x'_1 | w_2, x_1 \rangle_\epsilon = \mathrm{e}^{2i\pi s} \, \gamma_\epsilon(x_1, x'_1) \cdot (w'_2 - \bar{w}_2)^{-4\epsilon} \,, \tag{4.4}$$

where the notation was introduced for (see eq. (A.6))

$$\begin{aligned} \gamma_\epsilon(x_1, x'_1) &= a\big(s - ix_1 + \epsilon, s + ix'_1 + \epsilon\big)\, a\big(s + ix_1 + \epsilon, s - ix'_1 + \epsilon\big) \\ &= \frac{\mathrm{e}^{-2i\pi s}\, \Gamma(2\epsilon + i(x_1 - x'_1))\Gamma(2\epsilon + i(x'_1 - x_1))\Gamma^2(2s)}{\Gamma(\epsilon + s + ix_1)\Gamma(\epsilon + s - ix_1)\Gamma(\epsilon + s + ix'_1)\Gamma(\epsilon + s - ix'_1)} \,. \end{aligned} \tag{4.5}$$

Then, one substitutes (4.4) into (4.3), performs its Fourier transformation with a help of (A.4) and applies the identity

$$\lim_{\epsilon \to 0} \frac{\Gamma(2\epsilon - ix)\Gamma(2\epsilon + ix)}{\Gamma(4\epsilon)} = 2\pi\, \delta(x) \,. \tag{4.6}$$

to get the expression for $\langle p', x'_1 | p, x_1 \rangle \sim \delta(p - p')\delta(x_1 - x'_1)$, which matches the r.h.s. of eq. (3.8) at $N = 2$ for

$$\mu(x_1) = \frac{1}{2\pi} \left[\frac{\Gamma(s + ix_1)\Gamma(s - ix_1)}{\Gamma^2(2s)} \right]^2 . \tag{4.7}$$

This expression defines the integration measure in the SoV representation at $N = 2$. It is interesting to note that $\mu(x_1)$ coincides with the weight function for the continuous Hanh polynomials [21].

The calculation of the scalar product (4.1) for $N \geq 3$ is shown schematically in figure 6. At the first step, one uses the chain relation, figure 7, to integrate out the left- and rightmost vertices with the coordinates z_1 and z_N, respectively. This transformation replaces two pairs of lines by two vertical lines with the indices $\pm i(x_1 - x'_1)$ (see figure 6b) and brings the factor $\gamma_{\epsilon=0}(x_1, x'_1)$ defined in (4.5). Repeatedly applying the permutation identity, figure 3, we move one of the vertical lines horizontally through the diagram in the direction of another one until they meet and annihilate each other. In this way, we arrive at the diagram shown in figure 6c. In comparison with figure 6a, it has two vertices less and the parameters x_1 and x'_1 are interchanged. In this diagram there are already four vertices which can be integrated out using the chain relation, figure 7. Further simplification amounts to repetition of the steps just described. At each next step the number of vertices in the diagram is reduced and the $\boldsymbol{x}$-parameters got interchanged, $x_k \leftrightarrow x'_j$. Continuing this procedure, one obtains the diagram shown in figure 6d, in which $(N-1)$-elementary rhombuses are aligned along the vertical axis. It is accompanied by the factor $\prod_{j+k\leq N-1} \gamma_{\epsilon=0}(x_j, x'_k)$ defined in (4.5). Enumerating the rhombuses in figure 6d starting from the top, one finds that the kth rhombus coincides with the $N=2$ diagram shown in figure 8 upon replacement x_1 and x'_1 by x_{N-k} and x'_k, respectively. Based on the $N=2$ calculations, one should expect that the chain of rhombuses produces the contribution $\sim \prod_{k=1}^{N} \delta(x_{N-k} - x'_k)$. This turns out to be correct, but in order to see it one has to regularize the Feynman integrals according to (4.2) and carefully examine their limit as $\epsilon \to 0$. For $\epsilon \neq 0$ each elementary rhombus in figure 6d can be replaced by a single line with the index 4ε (see figure 6e). This brings the additional factor $\prod_{k=1}^{N-1} \gamma_\epsilon(x_k, x'_{N-k})$. Finally, one integrates out the remaining $(N-2)$-vertices using the chain relation, figure 7, and combines together all factors to find the following expression for the regularized scalar product (4.1)

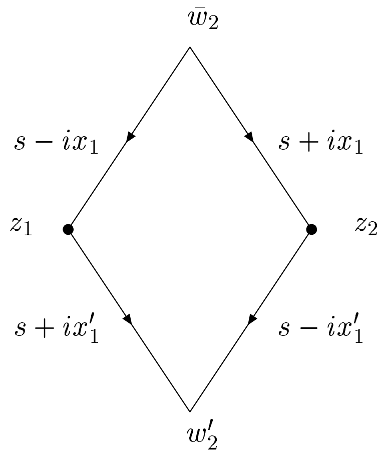

Figure 8: The scalar product of two pyramid diagrams at $N = 2$.

$$\langle w'_N, \boldsymbol{x}' | w_N, \boldsymbol{x}\rangle_\epsilon = \mathrm{e}^{i\pi N(N-1)s} \prod_{j+k\leq N-1} \gamma_{\epsilon=0}(x_j, x'_k) \prod_{k=1}^{N-1} \gamma_\epsilon(x_k, x'_{N-k}) \times$$
$$\times\, \mathrm{e}^{-i\pi(N-2)s} \frac{\Gamma^{N-2}(2s)\Gamma(\Delta)}{\Gamma^{N-1}(4\epsilon)} (w'_N - \bar{w}_N)^{-\Delta}, \tag{4.8}$$

where $\Delta = 4\epsilon(N-1) - 2s(N-2)$.

One substitutes (4.8) into (4.3), performs its Fourier transformation with a help of (A.4) and applies (4.6) to get after some algebra

$$\langle p', \boldsymbol{x}' | p, \boldsymbol{x}\rangle = (2\pi)^{N-1}\, \delta(p - p') \prod_{k=1}^{N-1} \delta(x_k - x'_{N-k}) \times$$
$$\times \Gamma^{N^2}(2s) \frac{\prod_{j<k} \Gamma(i(x_k - x_j))\Gamma(-i(x_k - x_j))}{\prod_{k=1}^{N-1} [\Gamma(s + ix_k)\Gamma(s - ix_k)]^N}. \tag{4.9}$$

This relation agrees with eq. (3.8), but in distinction with the latter its r.h.s. is not symmetric in $\boldsymbol{x}$. Indeed, simplifying the Feynman diagrams in figures 6, we have tacitly assumed that the $\gamma_{\epsilon=0}$-factors entering the r.h.s. of (4.8) are finite functions of $\boldsymbol{x}$ and $\boldsymbol{x'}$. Taking into account (4.5), one finds that this is true provided that $x_j \neq x'_k$ with $j,\, k = 1,\ldots,N-1$ and $j+k \leq N-1$.[4] Under these conditions, only one term survives in the r.h.s. of (3.8) and it has the same form as (4.9).

Matching (4.9) into (3.8) one obtains the integration measure for $N \geq 3$

$$\mu(\boldsymbol{x}) = c_N \prod_{\substack{j,k=1\\ j<k}}^{N-1} (x_k - x_j)\, \sinh(\pi(x_k - x_j)) \prod_{k=1}^{N-1} \left[\Gamma(s+ix_k)\Gamma(s-ix_k)\right]^N , \tag{4.10}$$

with the normalization factor $c_N = [\Gamma^{N^2}(2s)(2\pi)^{N-1}(N-1)!\pi^{(N-1)(N-2)/2}]^{-1}$. The following comments are in order.

In the SoV representation, the measure (4.10) is a semi-positive definite function on the space of real $\boldsymbol{x}$-variables. It vanishes at the hyperplanes $x_j = x_k$. After analytical continuation to complex $\boldsymbol{x}$, $\mu(\boldsymbol{x})$ becomes a meromorphic function of x_k $(k = 1,\ldots,N-1)$ with the Nth order poles located along the imaginary axis at $x_k = \pm i(s+n)$ with $n \in \mathbb{N}$. The measure decreases exponentially fast when x_k goes to infinity along the real axis and the remaining $\boldsymbol{x}$-variables take finite values

$$\mu(\boldsymbol{x}) \sim \mathrm{e}^{-2\pi|x_k|}\, x_k^{2(Ns-1)} , \tag{4.11}$$

as $x_k \to \infty$ and $\operatorname{Im} x_k = $ fixed. One verifies that the obtained expression for the measure, eq. (4.10), satisfies the functional relation (3.13).

The transition function $U_{p,\boldsymbol{x}}(z)$, eqs. (3.28) and (3.29), satisfies the completeness condition

$$\int_0^\infty dp \int_{\mathbb{R}^{N-1}} d^{N-1}\boldsymbol{x}\, \mu(\boldsymbol{x})\, \left(U_{p,\boldsymbol{x}}(w_1,\ldots,w_N)\right)^*\, U_{p,\boldsymbol{x}}(z_1,\ldots,z_N) = \mathbb{K}(z;w) , \tag{4.12}$$

where $\int d^{N-1}\boldsymbol{x} \equiv \prod_{k=1}^{N-1}\int_{-\infty}^{\infty} dx_k$ and $\mathbb{K}(z;w)$ is the reproducing kernel (kernel of the unity operator) on the quantum space of the model $\mathcal{V}_N$ (see eq. (A.7))

$$\left[\mathbb{K}\cdot\Psi\right](z_1,\ldots,z_N) = \int \mathcal{D}^N w \prod_{k=1}^{N} \frac{\mathrm{e}^{i\pi s}}{(z_k - \bar{w}_k)^{2s}} \Psi(w_1,\ldots,w_N) = \Psi(z_1,\ldots,z_N) , \tag{4.13}$$

with $\Psi(z_1,\ldots,z_N)$ holomorphic in the upper half-plane. The relation (4.12) is verified at $N=2$ by an explicit calculation in appendix A. For $N \geq 3$ the calculation is more involved and will be presented elsewhere.

[4] To restore the missing terms in the r.h.s. of (4.9), one has to use the symmetry of the pyramid in order to rearrange its rows and repeat the same calculation.

5. Eigenfunctions in the SoV representation

Let us consider the properties of the eigenfunction in the SoV representation $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$ defined in eq. (3.3). Using (3.28), (4.1) and (A.2), one can rewrite (3.3) as

$$\langle w_N, \boldsymbol{x}|\Psi_{\boldsymbol{q},p}\rangle = \int \mathcal{D}^N z\, (U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N))^* \Psi_{\boldsymbol{q},p}(\vec{z}) = \mathrm{e}^{ipw_N}\, \Phi_{\boldsymbol{q}}(\boldsymbol{x}) \cdot \frac{\theta(p)\, p^{-s(N-2)-1/2}}{\Gamma(2s)} \,. \tag{5.1}$$

As was shown in section 3.1, the function $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$ satisfies the multi-dimensional Baxter equation (3.12). In this section we shall construct the solutions to (3.12). The analysis is based on the relation between the transition function to the SoV representation, $U_{p,\boldsymbol{x}}(z)$, and the Baxter $\mathbb{Q}$-operator for the $\mathrm{SL}(2,\mathbb{R})$ spin chain.

By the definition [10], the $\mathbb{Q}$-operator acts on the quantum space of the model $\mathcal{V}_N$, depends on the spectral parameter u and satisfies the following conditions. It commutes with the transfer matrix $\widehat{t}_N(u)$ and with itself for different values of the spectral parameter, $[\widehat{t}_N(u), \mathbb{Q}(v)] = [\mathbb{Q}(u), \mathbb{Q}(v)] = 0$ and fulfil the operator Baxter equation

$$\widehat{t}_N(u)\, \mathbb{Q}(u) = (u+is)^N \mathbb{Q}(u+i) + (u-is)^N \mathbb{Q}(u-i)\,. \tag{5.2}$$

The eigenstates of the model (2.8) diagonalize the $\mathbb{Q}$-operator

$$\mathbb{Q}(u)\Psi_{p,\boldsymbol{q}}(z_1,\ldots,z_N) = Q_{\boldsymbol{q}}(u)\Psi_{p,\boldsymbol{q}}(z_1,\ldots,z_N)\,, \tag{5.3}$$

and the corresponding eigenvalues $Q_{\boldsymbol{q}}(u)$ satisfy the same equation (5.2). The Schrödinger equation (2.8) is equivalent to (5.3) and the energy $E_{\boldsymbol{q}}$ can be calculated as a logarithmic derivative of $Q_{\boldsymbol{q}}(u)$ at $u = \pm is$ [9, 22, 16].

The Baxter $\mathbb{Q}$-operator for the homogenous $\mathrm{SL}(2,\mathbb{R})$ spin magnet was constructed as an integral operator in ref. [16]

$$[\mathbb{Q}(u)\ \Psi]\,(z_1,\ldots,z_N) = \int \mathcal{D}^N w\, Q_u(z_1,\ldots,z_N;\bar{w}_1,\ldots,\bar{w}_N)\, \Psi(\bar{w}_1,\ldots,\bar{w}_N)\,. \tag{5.4}$$

and the kernel was calculated following the Pasquier–Gaudin approach [20] as

$$Q_u(\vec{z}\,;\vec{w}) = \mathrm{e}^{i\pi sN} \prod_{k=1}^{N} (z_k - \bar{w}_{k+1})^{-s+iu} (z_k - \bar{w}_k)^{-s-iu}\,, \tag{5.5}$$

where $w_{N+1} = w_1$. The conjugated operator $(\mathbb{Q}(u^*))^\dagger$ is defined in a similar way and its kernel is given by $(Q_{u^*}(\vec{w}\,;\vec{z}))^*$. Using (5.5) one can verify the following relation

$$(\mathbb{Q}(u^*))^\dagger = \mathbb{P}\,\mathbb{Q}(u) = \mathbb{Q}(u)\,\mathbb{P}\,, \tag{5.6}$$

where $\mathbb{P}$ is the operator of cyclic permutations of N particles, $\mathbb{P}^N = \mathbb{1}$,

$$\mathbb{P}\,\Psi_{\boldsymbol{q},p}(z_1,\ldots,z_{N-1},z_N) = \Psi_{\boldsymbol{q},p}(z_2,\ldots,z_N,z_1) = \mathrm{e}^{i\theta_{\boldsymbol{q}}}\, \Psi_{\boldsymbol{q},p}(z_1,\ldots,z_{N-1},z_N)\,. \tag{5.7}$$

Here in the second relation we took into account that the hamiltonian of the model, eq. (2.3), is invariant under the cyclic permutations of particles and, therefore, its eigenstates possess a definite value of the quasimomentum $\theta_{\boldsymbol{q}} = 2\pi k/N$ with $k \in \mathbb{N}$. It follows from (5.6) that the operators $(\mathbb{Q}(u^*))^\dagger$ and $\mathbb{Q}(u)$ share the common set of the eigenfunctions and their eigenvalues satisfy the relation

$$(Q_{\boldsymbol{q}}(u^*))^* = \mathrm{e}^{i\theta_{\boldsymbol{q}}}\, Q_{\boldsymbol{q}}(u)\,. \tag{5.8}$$

At $u = -is$ the kernel of the $\mathbb{Q}$-operator (5.5) coincides with the kernel of the unity operator (4.13) leading to

$$\mathbb{Q}(-is) = \mathbb{K}, \tag{5.9}$$

so that $Q_{\boldsymbol{q}}(-is) = 1$.

Notice that eq. (5.5) looks similar to the definition of the kernel of the Λ-operator, eq. (3.23), which enters into the expression for the SoV transformation (3.29). Namely, the expression for $Q_u(\vec{z}\,;\vec{w})\,\mathrm{e}^{-i\pi sN}$, eq. (5.5), coincides with the Y-function, eq. (3.21), and it differs from the Λ-function, eq. (3.23), by two factors, $(z_1-\bar{w}_1)^{-s-iu}$ and $(z_N-\bar{w}_1)^{-s+iu}$. This suggests that there should exist a relation between the transition function to the SoV representation $U_{\boldsymbol{x}}(\vec{z}\,;\bar{w}_N)$ and the kernel of the $\mathbb{Q}$-operator. For the $\mathrm{SL}(2,\mathbb{C})$ magnet such relation has been found in ref. [14].

As a starting point, one considers the following transformation $\Psi \to \Phi^\Omega$, the so-called separating map [23]

$$\begin{aligned}\Phi^\Omega(x_1,\ldots,x_{N-1}) &= \langle\Omega|\,\mathbb{Q}(x_1)\ldots\mathbb{Q}(x_{N-1})|\Psi\rangle \\ &= \int \mathcal{D}^N z_1 \ldots \int \mathcal{D}^N z_N\, (\Omega(\vec{z}_1))^* Q_{x_1}(\vec{z}_1;\vec{z}_2)\ldots Q_{x_{N-1}}(\vec{z}_{N-1};\vec{z}_N)\Psi(\vec{z}_N)\,,\end{aligned} \tag{5.10}$$

where $\Omega(z_1,\ldots,z_N)$ is an arbitrary function holomorphic on the upper-half plane. When applied to the eigenfunction of the model, $\Psi_{p,\boldsymbol{q}}(z_1,\ldots,z_N)$, it separates the variables for arbitrary $\Omega(\vec{z})$

$$\Phi^\Omega(x_1,\ldots,x_{N-1}) = Q_{\boldsymbol{q}}(x_1)\ldots Q_{\boldsymbol{q}}(x_{N-1})\langle\Omega|\Psi_{\boldsymbol{q},p}\rangle\,. \tag{5.11}$$

To reconstruct the eigenfunction $\Psi_{\boldsymbol{q},p}(z_1,\ldots,z_N)$ one has to invert (5.10) and define the inverse separating map $\Phi^\Omega \to \Psi$. Notice that for arbitrary $\Omega(\vec{z})$ the transformation (5.10) is not unitary and, therefore, it does not correspond to the SoV transformation (5.1). Let us demonstrate however that there exists special state $\Omega(\vec{z})$, for which the separating map (5.10) becomes unitary. In that case, (5.10) coincides with the SoV transformation (5.1) and the inverse separating map is defined by (3.1).

Following our diagrammatical approach, we use (5.5) and represent the kernel of the product of $(N-1)$ Baxter operators, $[\mathbb{Q}(x_1)\ldots\mathbb{Q}(x_{N-1})]\,(\vec{z}\,;\vec{w})$ as the Feynman diagram shown in figure 9 to the left. Notice that it contains as a subgraph the pyramid diagram corresponding to the transition function $U_{p,\boldsymbol{x}}(\vec{z};\bar{w}_N)$. It is possible to reduce the whole diagram to its subgraph as follows. One explores a freedom in choosing the $\vec{w}$-coordinates to put $\bar{w}_1 = \bar{w}_2$ in the upper row of the left diagram in figure 9. The two lines connecting the points $\bar{w}_1$ and $\bar{w}_2$ with the vertex y_1 merge into a single line with the index $2s$, which corresponds (up to a numerical factor) to the reproducing kernel $K(y_1,\bar{w}_1) = \mathrm{e}^{i\pi s}(y_1 - \bar{w}_1)^{-2s}$, defined in eqs. (A.7) and (4.13). Therefore, one can effectively remove the y_1-integration and put $y_1 = w_1$ instead. At the diagrammatical level this is equivalent to shrinking into a point the two lines connecting $\bar{w}_1$ and $\bar{w}_2$ with y_1. In a similar manner,

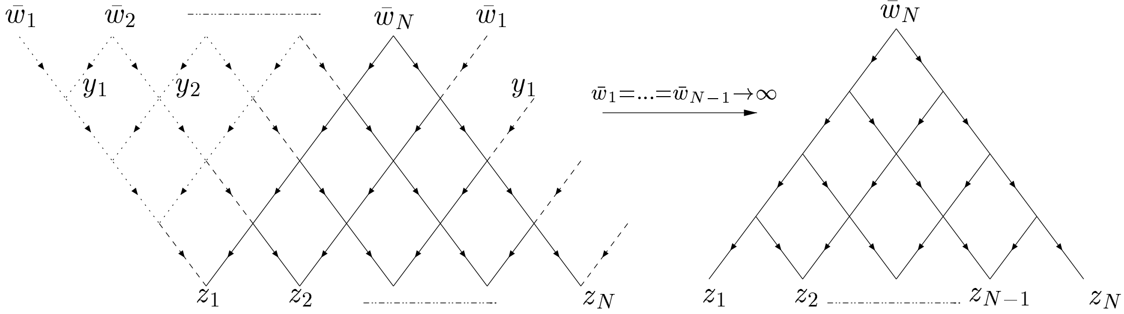

Figure 9: Reduction of the product of $(N-1)$ Baxter $\mathbb{Q}$-operators to the pyramid diagram. The dotted lines shrink into a point at $\bar w_1 = \bar w_2 = \ldots = \bar w_{N-1}$. The dashed lines are reduced to a power of $\bar w_1$ in the limit $w_1 \to \infty$.

choosing $\bar w_1 = \ldots = \bar w_{N-1}$ in the upper row of the left diagram in figure 9, one creates an avalanches of simplifications throughout the diagram which removes $2(k-1)$ lines in the kth row shown there by the dotted lines. The resulting diagram still contains $2(N-1)$ additional lines (the dashed lines in the figure 9), which connect the point w_1 with the vertices along the boundary of the pyramid. For $w_1 \to \infty$ their product scales as $\bar w_1^{-2s(N-1)}$. To get rid of the additional lines one multiplies the whole diagram by $\bar w_1^{2s(N-1)}$ and sends $w_1 \to \infty$.

In this way, we arrive at the following relation

$$U_{\boldsymbol{x}}(\vec z\,; \bar w_N) = i^{-(N-1)(N+2)s} \lim_{w_1 \to \infty} \bar w_1^{2s(N-1)} \left[\mathbb{Q}(x_1) \ldots \mathbb{Q}(x_{N-1})\right](\vec z\,; \bar w_1, \ldots, \bar w_1, \bar w_N)\,. \tag{5.12}$$

It can be rewritten in an operator form by introducing the special state

$$\Omega_{\bar w_0, \bar w_N}(z_1, \ldots, z_N) = i^{-(N-1)(N+2)s} \prod_{k=1}^{N-1} \frac{\mathrm{e}^{i\pi s}\, \bar w_0^{2s}}{(z_k - \bar w_0)^{2s}} \cdot \frac{\mathrm{e}^{i\pi s}}{(z_N - \bar w_N)^{2s}}\,, \tag{5.13}$$

which depends on two complex parameters $\bar w_0$ and $\bar w_N$. This state is normalizable with respect to the $\mathrm{SL}(2,\mathbb{R})$ scalar product

$$\langle \Omega_{\bar w_0', \bar w_N'} | \Omega_{\bar w_0, \bar w_N} \rangle = \mathrm{e}^{i\pi N s} (\bar w_0^{-1} - w_0'^{\,-1})^{-2s(N-1)} (w_N' - \bar w_N)^{-2s} \tag{5.14}$$

and, therefore, it belongs to the quantum space of the model. Then, one finds from (5.12) the following relation between the unitary transformation to the SoV representation and the product of $(N-1)$ Baxter $\mathbb{Q}$-operators

$$U_{\boldsymbol{x}}(\vec z\,; \bar w_N) = \lim_{\bar w_0 \to \infty} \langle \vec z | \, \mathbb{Q}(x_1) \ldots \mathbb{Q}(x_{N-1}) | \Omega_{\bar w_0, \bar w_N} \rangle\,, \tag{5.15}$$

where it is implied that one has to evaluate the matrix element and take the limit $\bar w_0 \to \infty$ afterwards. Substituting this relation into (5.1) and taking into account (5.6), one obtains

$$\begin{aligned} \langle w_N, \boldsymbol{x} | \Psi_{\boldsymbol{q},p} \rangle &= \lim_{\bar w_0 \to \infty} \langle \Omega_{\bar w_0, \bar w_N} | \, \mathbb{Q}(x_1) \ldots \mathbb{Q}(x_{N-1}) \mathbb{P}^{N-1} | \Psi_{\boldsymbol{q},p} \rangle \\ &= c_{\boldsymbol{q}}(p) \cdot \mathrm{e}^{ipw_N}\, Q_{\boldsymbol{q}}(x_1) \ldots Q_{\boldsymbol{q}}(x_{N-1})\,, \end{aligned} \tag{5.16}$$

where $c_{\boldsymbol{q}}(p) = \mathrm{e}^{-i\theta_{\boldsymbol{q}}} \lim_{\bar w_0 \to \infty} \langle \Omega_{\bar w_0, \bar w_N = 0} | \Psi_{\boldsymbol{q},p} \rangle$. The p-dependence of $c_{\boldsymbol{q}}(p)$ follows from (5.1) as $c_{\boldsymbol{q}}(p) = c_{\boldsymbol{q}} \cdot p^{-s(N-2)-1/2}/\Gamma(2s)$. Matching the second relation in (5.16) into the r.h.s. of (5.1) one obtains that the wave function in the Separated Variables is given by the product of $(N-1)$ eigenvalues of the Baxter operator (5.3) evaluated for real $u = x_k$

$$\Phi_{\boldsymbol{q}}(\boldsymbol{x}) = c_{\boldsymbol{q}}\, Q_{\boldsymbol{q}}(x_1)\dots Q_{\boldsymbol{q}}(x_{N-1})\,, \tag{5.17}$$

with the factor $c_{\boldsymbol{q}}$ depending on the normalization of the eigenstates $\Psi_{\boldsymbol{q},p}$.

It is well-known [16, 9, 22] that the eigenvalues of the Baxter operator for the quantum $\mathrm{SL}(2,\mathbb{R})$ magnet, $Q_{\boldsymbol{q}}(u)$, are polynomials in u of degree $h \in \mathbb{N}$ defined by the total spin of the system

$$\vec{S}^2\,\Psi_{\boldsymbol{q},p}(\vec{z}) = (h+Ns)(h+Ns-1)\,\Psi_{\boldsymbol{q},p}(\vec{z})\,, \tag{5.18}$$

or equivalently $q_2 = -(h+Ns)(h+Ns-1) - Ns(s-1)$ in eq. (2.20). Then, one finds from (5.17) that the wave function $\Phi_{\boldsymbol{q}}(\boldsymbol{x})$ is given by the product of $(N-1)$ polynomials in the separated variables. The eigenfunctions of the model $\Psi_{\boldsymbol{q},p}(\vec{z})$ are orthogonal to each other with respect to the $\mathrm{SL}(2,\mathbb{R})$ scalar product (2.9). Going over to the SoV representation (3.1) and using (3.8), one can write the orthogonality condition as

$$\langle \Psi_{\boldsymbol{q}',p'}|\Psi_{\boldsymbol{q},p}\rangle = \langle \Phi_{\boldsymbol{q}'}|\Phi_{\boldsymbol{q}}\rangle_{\mathrm{SoV}}\,\delta(p-p') = \delta(p-p')\,\delta_{\boldsymbol{q},\boldsymbol{q}'}\,, \tag{5.19}$$

where we took into account that the spectrum of the integrals of motion for the $\mathrm{SL}(2,\mathbb{R})$ magnet is discrete [22, 6]. Here the notation was introduced for the scalar product in the SoV representation

$$\langle \Phi_{\boldsymbol{q}'}|\Phi_{\boldsymbol{q}}\rangle_{\mathrm{SoV}} = \int_{\mathbb{R}^{N-1}} d^{N-1}\boldsymbol{x}\,\mu(\boldsymbol{x})\big(\Phi_{\boldsymbol{q}'}(x_1,\dots,x_{N-1})\big)^*\Phi_{\boldsymbol{q}}(x_1,\dots,x_{N-1})\,. \tag{5.20}$$

Together with (5.17) and (5.8) this leads to the orthogonality condition on the space of solutions to the Baxter equation (5.2) and (5.3)

$$\int_{\mathbb{R}^{N-1}} d^{N-1}\boldsymbol{x}\,\mu(\boldsymbol{x}) \prod_{k=1}^{N-1} Q_{\boldsymbol{q}}(x_k)\, Q_{\boldsymbol{q}'}(x_k) \sim \delta_{\boldsymbol{q},\boldsymbol{q}'}\,. \tag{5.21}$$

At $N=2$ this condition alone allows us to obtain the solution to the Baxter equation. Given that the integration measure at $N=2$, eq. (4.7), coincides with the weight function for the continuous Hanh orthogonal polynomials [21], one finds that $Q_{\boldsymbol{q}}(x)$ equals the same polynomial

$$Q_h(x) = {}_3F_2\left(\left.\begin{matrix} 4s+h-1, -h, s+ix \\ 2s, 2s\end{matrix}\,\right|\,1\right), \tag{5.22}$$

with nonnegative integer h defined in (5.18). This result is in agreement with the previous calculations [9, 22, 16, 6]. For higher N the solutions to the Baxter equation (5.2) can be expanded over the $N=2$ solutions (5.22) with the coefficients satisfying the N-term recurrence relations [22, 6, 7].

Substituting (5.17) into (3.1), one obtains the integral representation for the eigenfunctions of the model $\Psi_{\boldsymbol{q},p}(\vec{z})$. As shown in appendix B, the resulting expression for $\Psi_{\boldsymbol{q},p}(\vec{z})$ coincides with the well-known highest-weight representation for the eigenstates in the Algebraic Bethe Ansatz approach [1, 3]

$$\Psi_{\boldsymbol{q},p}(\vec{z}) = B_N(\lambda_1)\dots B_N(\lambda_h)\,\Omega_p(\vec{z})\,. \tag{5.23}$$

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Here the operator $B_N(u)$ is the off-diagonal element of the monodromy matrix (2.13) and the state $\Omega_p(\vec{z})$ is defined as

$$\Omega_p\left(\vec{z}\right) = \frac{p^{2s-1}\,\mathrm{e}^{i\pi Ns}}{\Gamma(2s)} \int_{\mathrm{Im}\,w>0} \mathcal{D}w\ \mathrm{e}^{ipw} \prod_{k=1}^{N} (z_k - \bar{w})^{-2s} . \tag{5.24}$$

The parameters $\lambda_1, \ldots, \lambda_h$ are simple roots of the eigenvalues of the Baxter operator, $Q_{\boldsymbol{q}}(u) = \prod_{k=1}^{h}(u - \lambda_k)$. They satisfy the Bethe equations

$$\left(\frac{\lambda_k + is}{\lambda_k - is}\right)^N = \prod_{\substack{j=1 \\ j\neq k}}^{h} \frac{\lambda_k - \lambda_j - i}{\lambda_k - \lambda_j + i}\,, \tag{5.25}$$

which follow from the Baxter equation (5.2) for $Q_{\boldsymbol{q}}(u)$. The fact that the two different representations for the eigenstates, eqs. (3.1) and (5.23), coincide implies that the Algebraic Bethe Ansatz is complete. We recall that the quantum space of the $\mathrm{SL}(2,\mathbb{R})$ magnet is infinite-dimensional for an arbitrary number of sites N and, as a consequence, the well-known combinatorial completeness analysis of the Bethe states (see ref. [24] and references therein) is not applicable in that case.

6. Conclusions

In this paper we have studied the spectral problem for the quantum $\mathrm{SL}(2,\mathbb{R})$ spin magnet within the framework of the Quantum Inverse Scattering Method. This model has previously emerged in high-energy QCD as describing the spectrum of anomalous dimensions of composite high-twist operators.

The central point of our analysis was the construction of the representation of the Separated Variables for the $\mathrm{SL}(2,\mathbb{R})$ spin magnet. Following the Sklyanin's approach, we presented a general method for constructing the unitary transformation to the SoV representation. It allowed us to obtain the integral representation for the eigenfunctions of the model and calculate explicitly the integration measure defining the scalar product in the SoV representation. We demonstrated that the language of Feynman diagrams becomes extremely useful in establishing various properties of the model. In particular, we found that the kernel of the unitary transformation to the SoV representation takes the factorized form (3.29). In terms of Feynman diagrams this implies that the kernel can be described by the pyramid diagram shown in figure 5. Notice that the same diagram has been previously encountered in the construction of the SoV representation for the $\mathrm{SL}(2,\mathbb{C})$ spin magnet in ref. [14, 15]. The approach described in this paper can be applied to other quantum integrable models like the periodic Toda chain [20, 13] and the DST model [25], which represent degenerated cases of the $\mathrm{SL}(2,\mathbb{R})$ spin chain.

Remarkably enough, many nontrivial properties of the SoV representation can be deduced in our approach from a few elementary identities between Feynman diagrams like the chain and permutation identities shown in figures 7 and 3, respectively. We found that the kernel of the unitary transformation to the SoV representation is given by the product of

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the Baxter $\mathbb{Q}$-operators projected onto a special reference state. This allowed us to establish the relation between the wave functions in the separated variables and the eigenvalues of the $\mathbb{Q}$-operator for the $\mathrm{SL}(2,\mathbb{R})$ spin magnet. Using the well-known fact that the latter are polynomials in the spectral parameter, we demonstrated the equivalence between two different expressions for the eigenstates obtained within the Algebraic Bethe Ansatz and the SoV method.

So far we have assumed that the spin chain was homogeneous. Another advantage of our method is that it can be easily extended to the case of the inhomogeneous closed spin chain. For the $\mathrm{SL}(2,\mathbb{R})$ chain with the spin in the kth site equals s_k, the unitary transformation to the SoV representation is defined by *the same* pyramid diagram (see figure 5) with the α- and β-indices modified in the following way. For $s_k \neq s_j$, the indices take different values for the different lines in the same row. If one denotes by α_{nk} and β_{nk} the indices carried by the kth pair of adjacent lines in the nth row from the bottom, $(1 \leq n \leq N-1$ and $1 \leq k \leq N-n)$, then $\alpha_{nk} = s_k - ix_n$ and $\beta_{nk} = s_{n+k} + ix_n$. In addition, in order to preserve the $\mathrm{SL}(2,\mathbb{R})$ invariance of integrals, one has to modify the integration measure (2.7) in all vertices of the diagram. Namely, one has to replace $(2\,\mathrm{Im}\,w_{nk})^{2s-2} \to (2\,\mathrm{Im}\,w_{nk})^{\alpha_{nk}+\beta_{nk}-2}$ in the integration measure (2.7) at the vertex w_{nk}, to which the lines α_{nk} and β_{nk} are attached.

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A. Feynman diagram technique

In this Appendix we collect some useful formulae, which were used to prove the permutation identity shown in figure 3 and to calculate of the integration measure (4.10).

To evaluate the $\mathrm{SL}(2,\mathbb{R})$ integrals we use the identity

$$\int_{\mathrm{Im}\,w>0} \mathcal{D}w\, \mathrm{e}^{ipw-ip'\bar{w}} = \delta(p-p')\, p^{1-2s} \cdot \Gamma(2s)\,, \tag{A.1}$$

with $\mathcal{D}w$ defined in (2.7), $\bar{w} = w^*$ and $p > 0$. The momentum representation for function $\Psi(z)$ holomorphic in the upper-half plane is defined as

$$\Psi(z) = \int_0^\infty dp\ \mathrm{e}^{ipz}\, \widetilde{\Psi}(p)\,, \qquad \widetilde{\Psi}(p) = \frac{\theta(p)p^{2s-1}}{\Gamma(2s)} \int_{\mathrm{Im}\,z>0} \mathcal{D}z\ \mathrm{e}^{-ip\bar{z}}\, \Psi(z)\,. \tag{A.2}$$

Applying the integral representation for the propagators

$$\frac{1}{(z-\bar{w})^\alpha} = \frac{\mathrm{e}^{-i\pi\alpha/2}}{\Gamma(\alpha)} \int_0^\infty dp\ \mathrm{e}^{ip\,(z-\bar{w})}\ p^{\alpha-1}\,, \tag{A.3}$$

which is valid for $\mathrm{Im}(z-\bar{w})>0$, one finds

$$\int_{\mathrm{Im}\,w>0} \mathcal{D}w\, \frac{\mathrm{e}^{ipw}}{(z-\bar{w})^{\alpha}} = \theta(p)\, p^{\alpha-2s}\, \mathrm{e}^{ipz}\cdot \mathrm{e}^{-i\pi\alpha/2}\,\frac{\Gamma(2s)}{\Gamma(\alpha)}\,, \tag{A.4}$$

$$\int_{\mathrm{Im}\,w>0} \mathcal{D}w\,(z-\bar{w})^{-\alpha}(w-\bar{v})^{-\beta} = a(\alpha,\beta)\,(z-\bar{v})^{-\alpha-\beta+2s}\,. \tag{A.5}$$

Here the notation was introduced for

$$a(\alpha,\beta) = \mathrm{e}^{-i\pi s}\,\frac{\Gamma(\alpha+\beta-2s)\Gamma(2s)}{\Gamma(\alpha)\Gamma(\beta)}\,. \tag{A.6}$$

Eq.(A.5) is the "chain relation" shown in figure 7. From eq.(A.4) and (A.2) one finds that

$$\int_{\mathrm{Im}\,w>0} \mathcal{D}w\,\frac{\mathrm{e}^{i\pi s}}{(z-\bar{w})^{2s}}\Psi(w) = \Psi(z)\,. \tag{A.7}$$

Using the Feynman parameterization

$$x_1^{-\alpha}x_2^{-\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 d\tau\tau^{\alpha-1}(1-\tau)^{\beta-1}[\tau x_1+(1-\tau)x_2]^{-\alpha-\beta}$$

and applying (A.7), one finds that for $\alpha+\beta=2s$ [16]

$$\int_{\mathrm{Im}\,w>0} \mathcal{D}w\,\frac{\mathrm{e}^{i\pi s}}{(z_1-\bar{w})^{\alpha}(z_2-\bar{w})^{\beta}}\Psi(w) = \frac{\Gamma(2s)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 d\tau\tau^{\alpha-1}(1-\tau)^{\beta-1}\Psi(\tau z_1+(1-\tau)z_2)\,. \tag{A.8}$$

The permutation relation shown in the figure 3 involves the Feynman integral

$$I(z,\bar{v}\,;x_1,x_2) = \int_{\mathrm{Im}\,w>0} \mathcal{D}w\frac{1}{(w-\bar{v}_1)^{\alpha_1}(w-\bar{v}_2)^{\beta_1}(z_1-\bar{w})^{\beta_2}(z_2-\bar{w})^{\alpha_2}}\,, \tag{A.9}$$

with $\alpha_k = s-ix_k$ and $\beta_k = s+ix_k$. One applies (A.8) for $\Psi(w)=(w-\bar{v}_1)^{-\alpha_1}(w-\bar{v}_2)^{-\beta_1}$ and calculates the integral in terms of ${}_2F_1$-hypergeometric function as

$$I(z,\bar{v}\,;x_1,x_2) = \mathrm{e}^{-i\pi s}\,(z_2-\bar{v}_2)^{-\alpha_2}(z_1-\bar{v}_1)^{-\alpha_1}\,(z_1-\bar{v}_2)^{\alpha_1+\alpha_2-2s}\,{}_2F_1\left(\alpha_1,\alpha_2;2s;\xi\right), \tag{A.10}$$

with $\xi=(z_1-z_2)(\bar{v}_1-\bar{v}_2)/[(z_1-\bar{v}_1)(z_2-\bar{v}_2)]$. From here, one gets for $x_1=x$ and $x_2=y$

$$(z_2-\bar{v}_2)^{i(x-y)}I(z,\bar{v}\,;x,y) = (z_1-\bar{v}_1)^{i(x-y)}I(z,\bar{v}\,;y,x)\,. \tag{A.11}$$

Multiplying the both sides of eq. (A.11) by $\bar{v}_2^{s-ix}$ and taking the limit $v_2\to\infty$, one obtains the reduced permutation relation shown in figure 4.

Finally, let us verify the completeness conditions (4.12) at $N=2$. We use (A.3) to rewrite the transition function $U_{p,x}(z_1,z_2)$ defined in (3.33) and (3.28) in the momentum representation as

$$U_{p,x}(z_1,z_2) = \frac{\mathrm{e}^{-i\pi s}\,\Gamma(2s)p^{1/2}}{\Gamma(s-ix)\Gamma(s+ix)}\int_0^\infty \frac{dp_1}{p_1^{1-s}}\int_0^\infty \frac{dp_2}{p_2^{1-s}}\left(\frac{p_2}{p_1}\right)^{ix}\mathrm{e}^{ip_1z_1+ip_2z_2}\,\delta(p_1+p_2-p)\,. \tag{A.12}$$

At $N = 2$ the pyramid diagram in figure 5 consists of two lines. The variables p_1 and p_2 have the meaning of the momenta that flow along these lines. Using similar expression for the conjugated function $(U_{p,x}(w_1, w_2))^*$ as a double integral over the momenta p'_1 and p'_2, one examines the product $\mu(x)U_{p,x}(z_1, z_2)(U_{p,x}(w_1, w_2))^*$ with the integration measure $\mu(x)$ given by (4.7). One notices that the x-dependence of the integration measure is compensated by the Γ-functions entering the r.h.s. of (A.12). As a result, the p- and x-integration in (4.12) can be easily performed leading to the following combination of the δ-functions

$$\delta(p_1 + p_2 - p'_1 - p'_2)\delta\left(\ln \frac{p_1 p'_2}{p_2 p'_1}\right) = \frac{p_1 p_2}{p_1 + p_2}\delta(p'_1 - p_1)\delta(p'_2 - p_2)\,. \tag{A.13}$$

Combining together all factors one evaluates the l.h.s. of (4.12) as

$$\int_0^\infty dp_1 \int_0^\infty dp_2\, \frac{(p_1 p_2)^{2s-1}}{\Gamma^2(2s)}\, \mathrm{e}^{ip_1(z_1-\bar{w}_1)+ip_2(z_2-\bar{w}_2)} = \frac{\mathrm{e}^{i\pi s}}{(z_1 - \bar{w}_1)^{2s}} \frac{\mathrm{e}^{i\pi s}}{(z_2 - \bar{w}_2)^{2s}}\,. \tag{A.14}$$

This expression coincides with the r.h.s. of (3.28) at $N = 2$.

B. Relation to the Algebraic Bethe Ansatz

In this appendix we demonstrate an equivalence between two different representations for the eigenstates of the $\mathrm{SL}(2,\mathbb{R})$ magnet, eqs. (3.1) and (5.23), obtained within the SoV method and the Algebraic Bethe Ansatz, respectively.

To begin with, one constructs the state equal to the product of the highest weights in N sites

$$\Omega(\vec{z}) = \prod_{k=1}^N \mathrm{e}^{i\pi s} z_k^{-2s}\,, \qquad S_k^+\,\Omega(\vec{z}) = 0\,. \tag{B.1}$$

It is analogous to a pseudovacuum state for compact spin magnets. This state annihilates the lower off-diagonal matrix element of the Lax operator (2.12) and, therefore, diagonalizes the transfer matrix (2.16)

$$\widehat{t}(u)\Omega(\vec{z}) = \left[(u + is)^N + (u - is)^N\right] \Omega(\vec{z})\,. \tag{B.2}$$

Due to the SL(2) invariance of the transfer matrix, $[\widehat{t}(u), \vec{S}] = 0$ with $\vec{S} = \sum_{k=1}^N \vec{S}_k$ being the total spin of the magnet, the state $\mathrm{e}^{\bar{w}S^-}\,\Omega(\vec{z}) = \Omega(z_1 - \bar{w}, \dots, z_N - \bar{w})$ also verifies (B.2).[5] Applying (A.2), one performs its Fourier transformation

$$\Omega_p(\vec{z}) = \frac{p^{2s-1}}{\Gamma(2s)} \int_{\mathrm{Im}\,w>0} \mathcal{D}w\ \mathrm{e}^{ipw}\, \Omega(z_1 - \bar{w}, \dots, z_N - \bar{w})\,. \tag{B.3}$$

The state $\Omega_p(\vec{z})$ carries the total momentum $p > 0$, diagonalizes the transfer matrix (B.2) and satisfies the following relations

$$iS^-\,\Omega_p(\vec{z}) = p\,\Omega_p(\vec{z})\,, \qquad \vec{S}^2\,\Omega_p(\vec{z}) = s(s-1)\Omega_p(\vec{z})\,. \tag{B.4}$$

[5]Notice that in distinction with (B.1), the state $\Omega(z_1 - \bar{w}, \dots, z_N - \bar{w})$ has a finite $\mathrm{SL}(2,\mathbb{R})$ norm for $\mathrm{Im}\,\bar{w} < 0$.

Its $\mathrm{SL}(2,\mathbb{R})$ norm can be easily calculated from (A.4) and (A.1) as

$$\langle \Omega_{p'}|\Omega_p\rangle = \int \mathcal{D}^N z \, \left(\Omega_{p'}(\vec{z})\right)^* \Omega_p(\vec{z}) = \delta(p-p')\frac{p^{2Ns-1}}{\Gamma(2Ns)} \,. \tag{B.5}$$

This implies that the state $\Omega_p(\vec{z})$ belongs to the quantum space of the model. Then, it follows from eqs. (B.2) and (B.4) that, in virtue of complete integrability of the model, $\Omega_p(\vec{z})$ is an eigenstate of the hamiltonian (2.3). Its total $\mathrm{SL}(2,\mathbb{R})$ spin can be found from (B.4) and (5.18) to be $h=0$. According to (2.4) and (2.3), the energy of this state is $E_{\boldsymbol{q}}=0$ since $(J_{n,n+1}-2s)\Omega_p(\vec{z})=0$ for $n=1,\dots,N$. The values of the corresponding integrals of motion $\boldsymbol{q}$ can be found by matching (B.2) into (2.18).

Let us transform the state $\Omega_p(\vec{z})$ into the SoV representation. One finds from (5.16)

$$\langle w_N, \boldsymbol{x}|\Omega_p\rangle = \lim_{w_0\to\infty} \langle \Omega_{\bar{w}_0,\bar{w}_N}|\, \mathbb{Q}(x_1)\dots\mathbb{Q}(x_{N-1})|\Omega_p\rangle \,, \tag{B.6}$$

since $\mathbb{P}\,\Omega_p(\vec{z})=\Omega_p(\vec{z})$ (see (5.7) and (B.1)). Notice that $\Omega(z_1-\bar{w},\dots,z_N-\bar{w})$ is proportional to the reproducing kernel (4.13) and its convolution with the kernel of the $\mathbb{Q}$-operator, eqs. (5.4) and (5.5), can be easily performed leading to $\mathbb{Q}(u)|\Omega_p\rangle=|\Omega_p\rangle$. In this way, one finds from (B.6) and (5.14)

$$\langle w_N, \boldsymbol{x}|\Omega_p\rangle = \mathrm{e}^{ipw_N}\,\frac{p^{2s-1}}{\Gamma(2s)}\,\mathrm{e}^{i\pi s(N-1)(N+4)/2} \,. \tag{B.7}$$

Together with the completeness condition (4.12) this leads to

$$\Omega_p(\vec{z}) = p^{Ns-1/2} i^{(N-1)(N+4)s} \int d^{N-1}\boldsymbol{x}\, \mu(\boldsymbol{x}) U_{p,\boldsymbol{x}}(\vec{z}) \,. \tag{B.8}$$

Substituting this relation into (B.5) and taking into account (3.8), one gets

$$\int_{\mathbb{R}^{N-1}} d^{N-1}\boldsymbol{x}\, \mu(\boldsymbol{x}) = \frac{1}{\Gamma(2Ns)} \,, \tag{B.9}$$

where the measure $\mu(\boldsymbol{x})$ is given by (4.10).

Let us apply the operator $B_N(\lambda_1)\dots B_N(\lambda_h)$ to the both sides of (B.8), with B_N being the off-diagonal matrix element of the monodromy operator (2.13) and λ_k some parameters. By the definition (3.4), the transition function $U_{p,\boldsymbol{x}}(\vec{z})$ diagonalizes the operator B_N

$$B_N(\lambda_1)\dots B_N(\lambda_h) U_{p,\boldsymbol{x}}(\vec{z}) = (-1)^{h(N-1)} p^h \prod_{j=1}^{N-1}\left\{\prod_{k=1}^{h}(x_j-\lambda_k)\right\} U_{p,\boldsymbol{x}}(\vec{z}) \,. \tag{B.10}$$

Let us choose $\lambda_1,\dots,\lambda_h$ to be roots of the eigenvalue of the Baxter $\mathbb{Q}$-operator, $Q_{\boldsymbol{q}}(\lambda_k)=0$, so that the expression in the parenthesis coincides with $Q_{\boldsymbol{q}}(x_j)$. Finally, one gets from (B.8) and (B.10)

$$B_N(\lambda_1)\dots B_N(\lambda_h)\Omega_p(\vec{z}) = c(p)\int d^{N-1}\boldsymbol{x}\,\mu(\boldsymbol{x})U_{p,\boldsymbol{x}}(\vec{z})\prod_{j=1}^{N-1} Q_{\boldsymbol{q}}(x_j) \tag{B.11}$$

with the normalization factor $c(p)=p^{Ns+h-1/2}(-1)^{h(N-1)}i^{(N-1)(N+4)s}$. The l.h.s. of (B.11) coincides with the well-known expression for the eigenstates in the Algebraic Bethe Ansatz, whereas the r.h.s. defines the integral representation for the same eigenstate in the SoV representation, eq. (3.1).

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